Area under the Excursion of Random Walk

Dissertation

zur Erlangung des akademischen Grades
Dr. rer. nat.

eingereicht an der
Mathematisch-Naturwissenschaftlich-Technischen Fakultät
der Universität Augsburg

von
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Augsburg, August 2019

Universität
Augsburg
University

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Mündliche Prüfung:	29. November 2019

Abstract

This work is devoted to the study of the area under a random process. In the second section we study the tail behaviour of the distribution of the area under the positive excursion of a random walk, which has negative drift and light-tailed increments. We determine the asymptotics for local probabilities for the area and prove a local central limit theorem for the duration of the excursion conditioned on the large values of its area. Next section is concentrated on the maximum of the excursion of a random walk with negative drift and light-tailed increments. More precisely, we determine the local asymptotics of the joint distribution of the length, maximum and the time at which this maximum is achieved. This result allows one to obtain a local central limit theorems for the length of the excursion conditioned on large values of the maximum. In the last section we continue studying tail behaviour of the distribution of the area under the positive excursion of a random walk which has a negative drift, but this time increments are heavy-tailed. We determine the asymptotics for tail probabilities for the area.

Zusammenfassung

In dieser Arbeit wird die Fläche unter der Exkursion einer Irrfahrt untersucht. Wir betrachten das asymptotische Verhalten des Tails der Fläche unter der positiven Exkursion einer Irrfahrt, wobei die Zuwächse "light-tailed"-verteilt sind und einen nicht positiven Erwartungswert haben. In dem zweiten Kapitel presentieren wir die lokale Asymptotiken für diese Fläche und beweisen den lokalen zentralen Grenzwertsatz für die Länge der Exkursion bedingt darauf, dass die Fläche sehr groß ist. In dem nächsten Kapitel untersuchen wir das Maximum der Exkursion der Irrfahrt, wobei die Zuwächse weiterhin "light-tailed"-verteilt sind, aber in diesem Fall einen negativen Erwartungswert besitzen. Wir finden die lokale Asymptotiken für die gemeinsame Verteilung von der Länge, dem Maximum und der Zeit in der dieser Maximum erreicht wird. Dieses Ergebnis ermöglicht uns den Beweis des lokalen Grenzwertsatzes für die Exkursionslänge diesmal bedingt darauf, dass der Maximum sehr groß ist. Anschließend vervollständigen wir die Untersuchung der Fläche unter der Exkursion einer Irrfahrt, wobei die Züwächse "heavy tailed"-verteilt sind und weiterhin den negativen Erwartungswert haben. Auch hier bestimmen wir die Asymptotiken für das Verhalten des Tails der Verteilung von der Fläche.

Danksagung

Nach der Vollendung dieser Doktorarbeit, richten sich meine Gedanken an all jene Menschen, welche mir mit Rat und Tat beiseite gestanden haben. Insbesondere möchte ich mich ganz herzlich bei meinem Doktorvater Herr Prof. Dr. Vitali Wachtel für die hervorragende Betreuung, unermessliche Geduld und ständige Hilfsbereitschaft bedanken. Der konstruktive Austausch und die regelmäßigen Gespräche auf fachlicher und persönlicher Ebene waren mir immer eine große Hilfe, haben mich inspiriert, sowie Motivation und Mut verliehen. Mein besonderer Dank gilt auch Frau Gerlinde Wolsleben, diejenige welche auf jede nicht mathematische Frage stets eine passende Antwort parat hat. Ihre Herzlichkeit und fabelhaftes Backtalent haben mir meinen Weg in dieser Zeit sehr versüßt. Bei meinen Büronachbarn Cédric Martin und Robert Nicholls bedanke ich mich für das kameradschaftliche Miteinander. Schließlich möchte ich mich ganz herzlich bei meiner Familie und meinen Freunden, die mir stets den Rücken gestärkt haben, bedanken. Meiner Mama und meiner Schwester möchte ich ganz besonders danken für ihre grenzenlose Unterstützung, Liebe und ihren Glaube an mich.
Diese Arbeit widme ich meinem Vater, er inspirierte mich diesen Weg zu gehen.

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1 Introduction

Queueing situations from daily life are obvious: customers queue up in front of the m cashiers in a supermarket; telephone callers waiting for a free line, aircrafts circling over the airport waiting for a free runway. More recent examples than these classical ones are number of problems connected with computer organisation or networks, or data transmission. There are a lot of examples that can be found in the finance and insurance sectors. The great diversity of queueing problems gives rise to an enormous variety of models, each with its specific features. Without attempting anything near a classification of all queueing situations, one can recognize the following relevant features for the description of a queue of reasonably simple structure:

- the input and the arrival process; i.e. how the customers arrive to the queue;
- the service facilities, i.e. how the system handles a given input stream;
- the queue discipline, i.e. the algorithm determining the order in which customers are served.

The description of these features can be quite complicated and lengthy. A convenient shorthand notation system was suggested by D.G. Kendall in 1953 and has to a large extent become standard since then. The symbolic notation covers some basic, but nevertheless important queueing systems which have the following characteristics:

- Customers arrive one at a time according to a renewal process in discrete or continuous time. That is, the intervals between successive arrivals of customers are independent, identically distributed (i.i.d.) with distribution A on $\mathbb{N}$ or $(0, \infty)$. We assume most often, that the customer 0 arrives at time 0. Thus, in T_n denotes the interval between the arrival time of customer n and $n+1$, the T_n are i.i.d. random variables with distribution A and the arrival instants are $0, T_0, T_0 + T_1, \dots$.
- The service times of different customers are i.i.d. and independent of the arrival process. We denote the distribution by B and the service time of customer n by U_n. Thus $U_0, U_1, \dots$ are i.i.d distributed with B and independent of the T_n.

In Kendal's notification, this queueing system is denoted by a string of type $\alpha/\beta/m$, where α refers to the form of the interarrival distribution, β to the form of the service time distribution and m is the number of services. In connection with a given queueing system, a great variety of stochastic pocesses and functionals arise. The main ones, that are the most interesting for us are the following three:

- Q_t- The queue length at time t.
- W_n- The waiting time of customer n, i.e. the time from arrival to the system until service starts.
- V_t- The workload in the system at time t, i.e. the total time the m servers have to work to clear the system.

There are a lot of studies about asymptotics for queueing systems. The tail behaviour of queueing process $\{Q(t), t \geq 0\}$, the workload process $\{W(t), t \geq 0\}$ or the busy period τ in standard systems has been well understood in both light- and heavy-tailed cases. But it is very suprising, that there are still many questions to be answered in reference to the tail behaviour of the area under a random process, more precisely the integral of the form:

$$I_f(T) := \int_0^T f(X(u))du,$$

where $\{X(t), t \geq 0\}$ is a stochastic process (for example, $X = Q$ or $X = W$), f is a deterministic function and T is either τ or deterministic (finite or infinite). In the paper of Kulik and Palmowski various open problems and conjectures are presented. Applications of such intergrals being used in many other fields than just queueing systems. The most known are for example financical mathematics, where integrals of the form $\int_0^\infty \exp(-X(u))du$ with X-a Levy process are used. Furthermore let $\{S(t), t \geq 0\}$ be a standard risk process, then integrals $\int_0^T \mathbf{1}_{\{S(u)<0\}}S(u)du$, where T is deterministic, can be used as possible risk measures. Other applications are coming from acrtuarial science, where very often regulated processes are considered and integrals from a regulation random mechanism are investigated. Moreover it is worth mentioning that areas of random walk excursions appear also in many combinatorical problems. In the paper of Winston K.J. and Kleitmann D.J. [39] the tournament score sequence problem is considered in a different way: lower and upper bounds of the number S_n of distinct score sequences are founded with help of corresponding random walks and the area under it. The paper [18] from C. Banderier and B. Gittenberger concerns the enumeration and asymptotics of the area below directed random walks on $\mathbb{N}$, with a finite set of jumps, so called latice paths. The key trick in their approach is the so-called "kernel method", which is a way of solving equations of the type $K(z,u)F(z,u) = A(z,u) + B(z,u)G(z)$, where F and G are the unknowns one wishes to determine. The kernel method consists in getting additional equations by plugging the roots of the "kernel" K in the initial equation, which is in general enough to solve the system. It is a nice result that the kernel method works also for a parameter like area. For a large class of walks, the authors give full asymptotics for the average area of excursion ("discrete" reflected Brownian bridge) and show that drift is not playing any role in this case. Another interesting application can be found in the paper [15] by P. Flajolet, P. Poblete and A. Viola. This paper presents moment analyses and characterizations of limit distributions for the construction cost of hash tables under the linear probing strategy. For full tables holds a limit law of so-called Airy type. The Airy distribution and its companion moment formula turn out to be a part of a ring of problems treated independently by a variety of methods and autors. In [15] there are five ranges of problems listed:

- construction cost in linear probing hashing,
- number of inversion in trees,
- connectivity in graphs,
- path length in trees,

- area of excursions.

The simplest type is the Bernoulli excursion defined by ± 1 steps (also called gambler's ruin sequence); Louchard [26], [27] established that the area of the Bernoulli excursion is asymptotically Airy distributed. One of the most classical combinatorical correspondences relates bijectively Bernoulli excursions and general Catalan trees. Under this correspondence, area of an excursion transforms into path length of the associated tree. Furthermore Takacs also identified the Airy distribution as a limiting distribution of the area under the Bernoulli excursion. His motivation was partially rooted in combinatorics. More precisely, he was interested in the ivestigation of the asymptotic number of random trees on n vertices with given total height, see Takacs [34], [35], [36] and Spencer [33]. And again using the well known one-to-one correspondence between random trees and random walk excursions it can be rewritten in a problem concerning the area under random walk path. Convergence for more general cases we can find in the works of Caravenna and Chaumont [5] and Sohier [32]. In particular, let $\{S_n;\, n \geq 1\}$ be a random walk with independent, identically distributed increments $\{X_k;\, k \geq 1\}$ and let τ be the first time when S_n is non-positive, i.e.,

$$\tau := \min\{n \geq 1 : S_n \leq 0\}.$$

The area under the trajectory $\{S_0, S_1, \dots, S_\tau\}$ we define as:

$$A_\tau := \sum_{k=1}^{\tau-1} S_k.$$

Let $\{S_n\}$ be first an integer-valued and centered random walk with finite second moments. Then it follows that the rescaled excursion of the random walk conditioned on $\tau = n+1$ converges weakly to the standard Brownian excursion $e(t), t \in [0,1]$. This implies that an appropriately rescaled area converges towards the corresponding functional of the Brownian excursion. More precisely,

$$\mathbb{P}(n^{-3/2}A_n \leq x | \tau = n+1) \longrightarrow \mathbb{P}\left(\int_0^1 e(t)dt \leq x\right), \quad x > 0. \tag{1.1}$$

This result allows one to find the asymptotic number of random trees on n vertices with the total height bounded by $xn^{3/2}$. But in order to find the number of trees with fixed total height, one needs a local version of (1.1). Moreover, such a result allows one to confirm the Kleitman-Winston conjecture mentioned above, see Takacs [34]. This conjecture was proved by Kim and Pittel [23] by deriving a uniform upper bound for probabilities $\mathbb{P}(A_n = a | \tau = n+1)$ in the case of a simple random walk. In the paper of Denisov, Kolb and Wachtel [8] this result was extended to a local limit theorem for the excursion area of an arbitrary random walk with zero mean and finite variance. They have shown

$$n^{3/2}\mathbb{P}(A_n = a | \tau = n+1) \longrightarrow C w_{ex}\left(\frac{a}{\sigma n^{3/2}}\right) \quad \text{as } n \to \infty, \tag{1.2}$$

here w_{ex} denotes the density of $\int_0^1 e(t)dt$.

The applications of the area under the excursion of random walk can be also used in other areas of science, for example in physics. Dobrushin and Hryniv discussed in their paper [11] statistical properties of random walks conditioned by fixing a large area under their paths and proved the functional central limit theorem (invariance principle) for these conditional distributions. From the point of view of statistical mechanics the studied problem is the problem of description of shapes of phase boundaries. From the mathematical point of view it is equivalent to the study of the asymptotical behaviour of the corresponding sequence of probability measures, describing the statistical properties of these boundaries. The simpliest variant of this problem occures in the one-dimensional Solid-On-Solid (SOS) model and has a nice probabilistic interpretation. The SOS-model consists of the interfaces without overhangs and therefore its configurations on of the horizontal length N are represented by sets of heights $\{r_i\}$, for $i = 0 \dots N$ with $r_0 = 0$. The energy of the configuration $R = \{r_i\}_{i=0}^N$ is determined by the Hamiltonian

$$\mathcal{H}_n(R) = \sum_{i=0}^{N-1} U(r_{i+1} - r_i),$$

where $U(\cdot)$ is a real valued function. There are many possible choices for this function. For simplicity the heights r_i are interger-valued. Let β be a positive parameter called an inverse temperature and assume that the partition function

$$Z_{N,\beta} = \sum_{r_1 \in Z} \dots \sum_{r_N \in Z} e^{-\beta \mathcal{H}_N(R)}$$

is finite, then the Gibbs probability distribution in the set of surfaces $\{r_i\}_{i=0}^N$ can be defined by

$$\mathbb{P}_{N,\beta}(R) = Z_{N,\beta}^{-1} e^{-\beta \mathcal{H}_N(R)}.$$

Rewriting the last expression in terms of jumps $k_i = r_i - r_{i-1}$, with $i = 1, \dots, n$, one can see that this Gibbs distribution coinsides with the probability distribution of random walk $r_0 = 0$ $r_j = \sum_{i=1}^j k_i$, $j \geq 1$, generated by the sequence of independent (interger-valued) jumps k_i having the same distribution

$$\mathbb{P}_\beta(k) = \frac{e^{-\beta U(k)}}{Z_\beta}, \quad \text{where} \quad Z_\beta = \sum_{k \in Z} e^{-\beta U(k)}.$$

Consequently, the main theorem of the paper [11] describes the statistical properties of the interface in 1D SOS-model with free right end and also studies the area under the interface. We take a closer look at the mathematical model. Let $X_1 \dots X_n$ be interger-valued independet random variables with the same probability distribution $\mathbb{P}(\cdot)$ with finite expectation $\mathbf{E}X_1 = \mu$ and variance $\sigma^2 \in (0; \infty)$. Let $\varphi(t)$ be the moment generating function of X_1, that is,

$$\varphi(t) := \mathbf{E}e^{tX_1},$$

and for some set of real λ:

$$L(\lambda) = \ln \varphi(\lambda) < \infty.$$

Consider the random walk $\{S_n, n \geq 1\}$. For any natural number n define a random polygonal function $s_n(t)$, $t \in [0,1]$:

$$s_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1},$$

where $[a]$ denotes the integer part of a real number a. Define

$$A_n = \sum_{j=0}^{n-1} S_k = \sum_{j=1}^{n} (n - j + 1)\, X_j,$$

the area under the graph of the piecewise constant function of $t \in [0, 1)$ which equals S_i on the interval $[k, k+1)$. The aim is to investigate the asymptotic behaviour of random paths $s_n(t)$ with fixed atypically large value of A_n. First, consider a new probability measure

$$\widehat{\mathbb{P}}(X_j \in dy) = \frac{e^{\lambda y}}{\varphi(\lambda)} \mathbb{P}(X_j \in dy).$$

Note that by the definition of $\widehat{\mathbb{P}}$, we have

$$\widehat{\mathbf{E}} X_j = L'(\lambda) = \frac{\varphi'(\lambda)}{\varphi(\lambda)}, \quad \widehat{\mathbf{Var}} X_j = L''(\lambda) = \frac{\varphi''(\lambda)}{\varphi(\lambda)} - \left(\frac{\varphi'(\lambda)}{\varphi(\lambda)}\right)^2 \quad j = 1, 2, \dots n.$$

Moreover let $L_{A_n}(\lambda)$ be the logarithmic moment generating function corresponding to the random variable A_n,

$$L_{A_n} = \ln \mathbf{E} e^{\frac{\lambda}{n} A_n} = \sum_{j=1}^{n} L\left(\left(\frac{n - j + 1}{n}\right)\lambda\right).$$

Denote also

$$L_{A_n,\infty} = \lim_{n\to\infty} L_{A_n}(\lambda) = \int_0^1 L(\lambda x) dx.$$

Consider any sequense nq_n of real numbers such that $n^2 q_n$ are interger and $q_n \to q \neq \frac{\mu}{2}$ in such a way, that

$$q_n - q = o\left(\frac{1}{\sqrt{n}}\right) \quad \text{as} \quad n \to \infty. \tag{1.3}$$

The sequence nq_n with some special restrictions must be fixed, then the random process

$$\theta_n(t) = (s_n(t) | A_n = n^2 q)$$

is well defined. Note that the variable A_n has the mean value $\mathbf{E} A_n = \mu n(n-1)/2$ and the variance $\mathbf{Var} A_n = \sigma^2 (n-1)(2n-1)/6$. Therefore the condition $2q \neq \mu$ corresponds to the situation of large values of A_n. Define

$$\psi_{\bar\lambda}(t) = -\frac{1}{\bar\lambda} \ln \frac{\varphi(\bar\lambda(1-t))}{\varphi(\bar\lambda)}$$

and consider normalized fluctuations of paths $\theta_n(t)$ around $n\psi_{\bar\lambda}(t)$,

$$\theta_n^*(t) = \frac{1}{\sqrt{n}} (\theta_n(t) - n\psi_{\bar\lambda}(t)).$$

Let μ_n^* denote the measure in the space $\mathbf{C}[0, 1]$ of continuous functions on the segment $[0, 1]$ induced by the probability distribution of the process $\theta_n^*(t)$. It is shown in [11], that the sequence of measures μ_n^* converges weakly to some Gaussian measure μ^* in $\mathbf{C}[0,1]$. The limiting measure μ^* coinsides with the conditional distribution of the random process $\bar{X}(t)$, $t \in [0,1]$, obtained by the integral transformation of the white noise dw_x,

$$\bar{X}(t) = \int_0^t \left(\widehat{\mathbf{Var}}(\bar{\lambda} - \bar{\lambda}x)\right)^{1/2} dw_x,$$

conditioned by fixing the value

$$A = \int_0^1 \bar{X}(t)dt = 0.$$

Together with (1.3) this statement implies the law of large numbers for $\theta_n(t)$: distributions of random processes $n^{-1}\theta_n(t)$ converge weakly in $\mathbf{C}[0,1]$ to the distribution concentrated on the (deterministic) function $\psi_{\bar{\lambda}}(t)$.

The paper [4] of A. A. Borovkov, O. J. Boxma and Z. Palmowski studies asymptotics of area under the workload process during one busy period when the service time distribution has a regularly varying tail. Moreover they investigate the case of a light-tailed service time distribution. They have shown that the occurence of a large area is related to the occurence of a large cycle maximum. These are known for the $GI/G/1$ queue with subexponential service time, see [2] or with light-tailed service time distribution, see [20]. The authors give the following asymptotics, as $x \to \infty$

$$\mathbb{P}(A_\tau > x) \sim \mathbb{P}(M_\tau > \sqrt{2|\mathbf{E}X_1|}x^{1/2}) \sim \mathbf{E}\tau\mathbb{P}(X_1 > \sqrt{2|\mathbf{E}X_1|}x^{1/2}),$$

where

$$M_\tau := \max_{n<\tau} S_n.$$

Behind this relation stays the strategy, that the random walk has one big jump at the beginning and then goes linearly down according to the law of large numbers. They also consider the case of light-tailed service times, proving that for $x \to \infty$

$$\log \mathbb{P}(A_\tau > x) \sim \sqrt{\frac{2\mu\varphi'(\lambda)}{\varphi'(\lambda) + \mu\varphi(\lambda)}} x^{1/2},$$

where $\varphi(t) := \mathbf{E}e^{tX_1}$, $t \geq 0$ the moment generating function of X_1 and the Cramér condition holds: $\varphi(\lambda) = 1$ for some $\lambda > 0$. But since $\mathbb{P}(M_\tau > y) \sim Ce^{-\lambda y}$ one derives a contradiction to the known results for random walks with two sided exponentially distributed increments, see Guillemin [19] and Kearney [22]. Our goal is to find the correct asymptotics for this case. So in the second section we study tail behaviour of the distribution of the area under the positive excursion of a random walk which has negative drift and light-tailed increments. We use another strategy based on the fact, that Duffy and Meyn have shown that the optimal path to a large area is a rescaling of the function

$$\psi(u) := \frac{1}{\lambda} \log \varphi(\lambda(1-u)), \quad u \in [0,1].$$

We give the asymptotics for local probabilities for the area without logarithmic scaling

$$\mathbb{P}(A = x) \sim \kappa x^{-3/4} e^{-\theta\sqrt{x}},$$

where $\theta := 2\lambda\sqrt{I}$ and $I := \int_0^1 \psi(u)du$. Then we prove a local central limit theorem for the duration of the excursion conditioned on the large values of its area for $x \to \infty$

$$\mathbb{P}(\tau = k | A_\tau = x) \longrightarrow \frac{1}{x^{1/4}\sqrt{2\pi\Delta^2}} \exp\left\{\frac{(k - mx^{1/2})^2}{2\Delta^2 x^{1/2}}\right\},$$

where $m = \left(\int_0^1 \psi(t)dt\right)^{-1/2}$. Considering the close connection between the maximum and the area under the excursion we study in the next section the maximum of the excursion of a random walk with negative drift and light-tailed increments. Now we can use the strategy, that the random walk goes up linearly with the rate $\widehat{a}$ and after reaching the level x, the random walk goes down with the standard rate a. We determine the local asymptotics of the joint distribution of the length, maximum and the time at which this maximum is achieved. If the Cramér condition holds and $\widehat{\sigma}^2 = \mathbf{E}[X_1^2 e^{\lambda X_1}] - \widehat{a}^2 \in (0, \infty)$, then we get

$$e^{\lambda x}\mathbb{P}(M_\tau = x, \theta_\tau = k, \tau = n+1) \approx C \exp\left\{-\frac{(x - k\widehat{a})^2}{2\widehat{\sigma}^2 k} - \frac{(x - a(n-k))^2}{2\sigma(n-k)}\right\},$$

where $\theta_\tau := \min\{n \geq 0 : S_n = M_\tau\}$. This result allows one to obtain a local central limit theorems for the length of the excursion conditioned on large values of the maximum. The fourth section is devoted to the study of the tail behaviour of the distribution of the area under the positive excursion of a random walk, which has a negative drift but this time with heavy-tailed increments. We have already seen the heavy-tailed asymptotics for $\mathbb{P}(A_\tau > x)$ by Borovkov, Boxma and Palmovski [4]. They considered the case when the increments of the random walk have a distribution with regularly varying tails using the traditional heavy-tailed one big jump strategy. But the class of regularly varying distributions does not include all subexponential distributions and excludes, in particular, log-normal distribution and Weibull distribution with parameter $\beta < 1$. So we extend this class of distributions and determine the asymptotics for tail probabilities for this area.

2 Area under excursion. Local asymptotics for the area under the random walk excursion

This section is the subject of the article [28] written in collaboration with
Vitali Wachtel
to appear in
Advances in Applied Probability.

In this section we study tail behaviour of the distribution of the area under the positive excursion of a random walk which has negative drift and light-tailed increments. We determine the asymptotics for local probabilities for the area and prove a local central limit theorem for the duration of the excursion conditioned on the large values of its area.

2.1 Introduction and statement of results

Let $\{S_n;\, n \geq 1\}$ be a random walk with independent, identically distributed increments $\{X_k;\, k \geq 1\}$ and let τ be the first time when S_n is non-positive, i.e.,

$$\tau := \min\{n \geq 1 : S_n \leq 0\}.$$

Define also the area under the trajectory $\{S_0, S_1, \dots, S_\tau\}$:

$$A_\tau := \sum_{k=1}^{\tau-1} S_k.$$

If the increments of the random walk have non-positive mean, then the random variables τ and A_τ are finite and we are interested in the tail behaviour of the area A_τ.
In the case of the driftless ($\mathbf{E}X_1 = 0$) random walk with finite variance $\sigma^2 := \mathbf{E}X_1^2 \in (0, \infty)$ one has a universal tail behaviour

$$\lim_{x\to\infty} x^{1/3}\mathbf{P}(A_\tau > x) = 2C_0\sigma^{1/3}\mathbf{E}\left(\int_0^1 e(t)dt\right)^{1/3}, \tag{2.1}$$

where $e(t)$ denotes the standard Brownian excursion and the constant C_0 is taken from the relation $\mathbf{P}(\tau = n) \sim C_0 n^{-3/2}$. Proposition 1 in Vysotsky [38] states that (2.1) holds for some particular classes of random walks. But one can easily see that the proof from [38] remains valid for all oscillating random walks with finite variance. Later we shall give an alternative proof of (2.1).
If the mean of X_1 is negative then the distribution of A_τ becomes sensitive to the tail behaviour of the increments. Borovkov, Boxma and Palmowski [4] have shown that if the tail of X_1 is a regularly varying function then, as $x \to \infty$,

$$\mathbf{P}(A_\tau > x) \sim \mathbf{P}\left(M_\tau > \sqrt{2|\mathbf{E}X_1|}x^{1/2}\right) \sim \mathbf{E}\tau\mathbf{P}\left(X_1 > \sqrt{2|\mathbf{E}X_1|}x^{1/2}\right), \tag{2.2}$$

where

$$M_\tau := \max_{n<\tau} S_n.$$

Behind this relation stays a simple heuristic explanation. In order to have a large area under the excursion the random walk has to make a large jump at the very beginning and then the random walks behaves according to the law of large numbers. More precisely, if the jump of size h appears, after which the random walk goes linearly down with the slope $-\mu$, where $\mu := |\mathbf{E}X_1|$, then the duration of the excursion will be of order h/μ. Consequently, the area will be of order $h^2/2\mu$. If we want the area to be of order x then the jump has to be of order $\sqrt{2\mu}x^{1/2}$. The same strategy is optimal for large values of M_τ. As a result, we have both asymptotic equivalences in (2.2).
This close connection between the maximum M_τ and the area A_τ is not valid for random walks with light tails. Let $\varphi(t)$ be the moment generating function of X_1, that is,

$$\varphi(t) := \mathbf{E}e^{tX_1}, \quad t \geq 0.$$

We shall consider random walks satisfying the Cramér condition:

$$\varphi(\lambda) = 1 \text{ for some } \lambda > 0. \tag{2.3}$$

Moreover, we shall assume that

$$\varphi'(\lambda) < \infty \quad \text{and} \quad \varphi''(\lambda) < \infty. \tag{2.4}$$

It is well-known that if (2.3) and (2.4) hold, then the most likely path to a large value of M_τ is piecewise linear. The random walk goes first up with the slope $\varphi'(\lambda)/\varphi(\lambda)$. After arrival at the desired level h, it goes down with the slope $-\mu$. If this path would be optimal for the area then

$$\mathbf{P}(A_\tau > x) \approx \mathbf{P}\left(M_\tau > \sqrt{\frac{2\mu\varphi'(\lambda)}{\varphi'(\lambda) + \mu\varphi(\lambda)}}x^{1/2}\right).$$

Since $\mathbf{P}(M_\tau > y) \sim Ce^{-\lambda y}$, one arrives at the contradiction to the known results for random walks with two-sided exponentially distributed increments, see Guillemin and Pinchon [19] and Kearney [22].
Duffy and Meyn [14] have shown, that the optimal path to a large area is a rescaling of the function

$$\psi(u) := -\frac{1}{\lambda}\log\varphi(\lambda(1-u)), \quad u \in [0,1]. \tag{2.5}$$

They have also shown that

$$\lim_{x\to\infty} \frac{1}{x^{1/2}} \log \mathbf{P}(A_\tau > x) = -\theta, \tag{2.6}$$

where

$$\theta := 2\lambda\sqrt{I} \quad \text{and} \quad I := \int_0^1 \psi(u)du.$$

Our purpose is to derive precise asymptotics for A_τ, without logarithmic scaling.

Theorem 2.1. *Assume that X_1 is integer valued and aperiodic. Assume also that (2.3) and (2.4) hold. Then there exists a positive constant $\varkappa$ such that*

$$\mathbf{P}(A_\tau = x) \sim \varkappa x^{-3/4}e^{-\theta\sqrt{x}}, \quad x \to \infty. \tag{2.7}$$

It is easy to see that (2.7) implies that

$$\mathbf{P}(A_\tau > x) \sim \frac{2\varkappa}{\theta} x^{-1/4} e^{-\theta\sqrt{x}}. \tag{2.8}$$

An analogue of this relation has been obtained by Guillemin and Pinchon [19] for an $M/M/1$ queue and by Kearney [22] for an $Geo/Geo/1$ queue.
Relation (2.8) confirms the conjecture in Kulik and Palmowski [25] for all integer valued random walks. Unfortunately, we do not know how to derive a version of (2.7) for non-lattice random walks. Moreover, we do not know how to derive (2.8) without local asymptotics. One can derive an upper bound for $\mathbf{P}(A_\tau > x)$ via the exponential Chebyshev inequality. This leads to

$$\mathbf{P}(A_\tau > x) \le C x^{1/4} e^{-\theta\sqrt{x}}. \tag{2.9}$$

For the proof of this estimate see Subsection 2.4. Comparing (2.8) and (2.9), we see that the Chebyshev inequality gives the right logarithmic rate of diveregence and that the error in (2.9) is of order $\sqrt{x}$. Such an error is quite standard for the exponential Chebyshev inequality. In the most classical situation of sums of i.i.d. random variables one has an error of order $\sqrt{n}$. In order to avoid this error and to obtain (2.7) we apply an appropriate exponential change of measure and analyse, under transformed measure, the asymptotic behavior of local probabilities for S_n and $A_n := \sum_{k=1}^n S_k$ conditioned on the event $\{\tau = n+1\}$. This approach allows one to obtain the following conditional limit for the duration of the excursion.

Theorem 2.2. *Under the assumptions of Theorem 2.1, there exists $\Delta^2 > 0$ such that*

$$\sup_k \left| x^{1/4}\mathbf{P}(\tau = k | A_\tau = x) - \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left\{ -\frac{(k - m x^{1/2})^2}{2\Delta^2 x^{1/2}} \right\} \right| \to 0, \quad x \to \infty,$$

where

$$m = \left(\int_0^1 \psi(t) dt \right)^{-1/2}.$$

2.2 Non-homogeneous exponential change of measure

Our approach to the derivation of the tail asymptotics for A_τ is based on a careful analysis of large deviation probabilities for the vector (A_n, S_n) conditioned on $\{\tau = n+1\}$. For every fixed n we shall perform the following non-homogeneous change of measure. Consider a new probability measure $\widehat{\mathbf{P}}$ such that, for every $j \le n$,

$$\widehat{\mathbf{P}}(X_j \in dy) = \frac{e^{u_{n,j} y}}{\varphi(u_{n,j})} \mathbf{P}(X_j \in dy), \tag{2.10}$$

where

$$u_{n,j} = \lambda \frac{(n-j+1)}{n}.$$

This non-homogeneous choice of transformation parameters $u_{n,j}$ can be easily explained by the fact that it corresponds to the exponential change of the distribution of A_n with parameter λ/n. Indeed,

$$\mathbf{E}e^{\frac{\lambda}{n}A_n} = \mathbf{E}e^{\frac{\lambda}{n}\sum_1^n(n-j+1)X_j} = \prod_{j=1}^n \varphi\left(\frac{n-j+1}{n}\lambda\right).$$

We have also the following relation between probabilities $\widehat{\mathbf{P}}$ and $\mathbf{P}$:

$$\mathbf{P}(A_n \in dx, S_n \in dy) = e^{-\lambda x/n}\prod_{j=1}^n \varphi(u_{n,j})\widehat{\mathbf{P}}(A_n \in dx, S_n \in dy) \tag{2.11}$$

and

$$\begin{aligned}&\mathbb{P}(A_n \in dx, S_n \in dy, \tau > n)\\ &\qquad = e^{-\lambda x/n}\prod_{j=1}^n \varphi(u_{n,j})\widehat{\mathbb{P}}(A_n \in dx, S_n \in dy, \tau > n).\end{aligned} \tag{2.12}$$

2.3 Simple properties of the change of measure

In this paragraph we shall collect some elementary properties of the measure change from (2.10). We first note that, by the definition of $\widehat{\mathbf{P}}$,

$$\widehat{\mathbf{E}}X_j = \frac{\varphi'(u_{n,j})}{\varphi(u_{n,j})}, \quad j = 1, 2, \dots, n.$$

This implies that if $j/n \to t \in [0,1]$ then

$$\widehat{\mathbf{E}}X_j \to \frac{\varphi'(\lambda(1-t))}{\varphi(\lambda(1-t))} \quad \text{and} \quad \frac{1}{n}\widehat{\mathbf{E}}S_j \to \int_0^t \frac{\varphi'(\lambda(1-u))}{\varphi(\lambda(1-u))}du = \psi(t).$$

More precisely, there exists a constant C such that, for all $j = 1, 2, \dots, n$,

$$\left|\widehat{\mathbf{E}}S_j - n\psi\left(\frac{j}{n}\right)\right| \le C. \tag{2.13}$$

This statement is a standard error estimate for the Riemann sum approximation of integrals of a function with bounded derivative. Furthermore,

$$\widehat{\mathbf{Var}}X_j = \frac{\varphi''(u_{n,j})}{\varphi(u_{n,j})} - \left(\frac{\varphi'(u_{n,j})}{\varphi(u_{n,j})}\right)^2$$

and consequently,

$$\frac{1}{n}\widehat{\mathbf{Var}}S_j \to \int_0^t \left(\frac{\varphi''(\lambda(1-u))}{\varphi(\lambda(1-u))} - \left(\frac{\varphi'(\lambda(1-u))}{\varphi(\lambda(1-u))}\right)^2\right)du.$$

From these asymptotics for the first two moments and from the Kolmogorov inequality we infer that

$$\sup_{t\in[0,1]}\left|\frac{S_{[nt]}}{n} - \psi(t)\right| \to 0, \quad \text{in } \widehat{\mathbf{P}} - \text{probability}.$$

Fix some $\gamma \in (0, 1/2)$. It is obvious that $\widehat{\mathbf{E}} X_j^3$ are uniformly bounded for $j \in [\gamma n, (1-\gamma)n]$. This implies that the sequence $\{X_j\}_{j=1}^n$ satisfies the Lindeberg condition. Therefore, we have the following version of the functional central limit theorem: the sequence of linear interplations

$$s_n(t) = n^{-1/2}\left(S_k + X_k(tn - k - 1) - n\psi(t)\right) \quad \text{for } t \in \left[\frac{k}{n}, \frac{k+1}{n}\right], k = 0, 1, \ldots, n-1$$

converges weakly on $C[0,1]$ towards a centered gaussian process $\{\xi(t); t \in [0,1]\}$ with independent increments and second moments

$$\mathbf{E}(\xi(t))^2 = \int_0^t \sigma^2(u)du,$$

where

$$\sigma^2(u) := \frac{\varphi''(\lambda(1-u))}{\varphi(\lambda(1-u))} - \left(\frac{\varphi'(\lambda(1-u))}{\varphi(\lambda(1-u))}\right)^2.$$

Convergence on $C[0,1]$ implies that

$$\left(\frac{S_{[nt]} - n\psi(t)}{\sqrt{n}}, \frac{A_{[nt]} - n^2 \int_0^t \psi(s)ds}{n^{3/2}}\right) \Rightarrow \left(\xi(t), \int_0^t \xi(s)ds\right), \quad t \in [0,1].$$

The limiting vector has a normal distribution with zero mean. We now compute the covariance of $\xi(t)$ and $\int_0^t \xi(s)ds$. Using the independence of the increments, one can easily get

$$\begin{aligned}
\mathbf{Cov}\left(\xi(t), \int_0^t \xi(s)ds\right) &= \int_0^t \mathbf{Cov}\left(\xi(t), \xi(s)\right) ds \\
&= \int_0^t \mathbf{Cov}\left(\xi(s) + \xi(t) - \xi(s), \xi(s)\right) ds \\
&= \int_0^t \mathbf{Cov}\left(\xi(s), \xi(s)\right) ds \\
&= \int_0^t \int_0^s \sigma^2(u)du\, ds = \int_0^t \sigma^2(u)(t-u)du.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbf{Cov}\left(\int_0^t \xi(s)ds, \int_0^t \xi(s)ds\right) &= \int_0^t \int_0^t \mathbf{Cov}\left(\xi(s_1), \xi(s_2)\right) ds_1 ds_2 \\
&= 2\int_0^t ds_1 \int_0^{s_1} \mathbf{Cov}\left(\xi(s_1), \xi(s_2)\right) ds_2 \\
&= 2\int_0^t ds_1 \int_0^{s_1} \left(\int_0^{s_2} \sigma^2(u)du\right) ds_2 \\
&= 2\int_0^t \int_0^{s_1} \sigma^2(u)(s_1 - u)ds_1\, du \\
&= \int_0^t \sigma^2(u)(t-u)^2 du.
\end{aligned}$$

Therefore, the density of $\left(\xi(t), \int_0^t \xi(s)ds\right)$ is given by

$$f_t(x,y) := \frac{1}{2\pi\sqrt{\det \Sigma_t}} \exp\left(-\frac{1}{2}(x,y)\Sigma_t^{-1}(x,y)^T\right), \tag{2.14}$$

with the covariance matrix

$$\Sigma_t = \begin{pmatrix} \int_0^t \sigma^2(u)du & \int_0^t \sigma^2(u)(t-u)du \\ \int_0^t \sigma^2(u)(t-u)du & \int_0^t \sigma^2(u)(t-u)^2du \end{pmatrix}. \tag{2.15}$$

2.4 Proof of the Chebyshev-type estimate (2.9)

Lemma 2.3. *As $n \to \infty$,*

$$\prod_{j=1}^{n} \varphi(u_{n,j}) = \exp\{-\lambda I n\}\left(1 + O(n^{-1})\right) \tag{2.16}$$

Proof.
It is obvious that (2.16) is equivalent to

$$\sum_{j=1}^{n} \log\varphi(u_{n,j}) = -\lambda I n + O(n^{-1}). \tag{2.17}$$

The sum on the left hand side of (2.17) can be written as follows:

$$\begin{aligned}
\sum_{j=1}^{n} \log\varphi\left(\lambda\frac{n-j+1}{n}\right) &= \sum_{j=1}^{n} \log\varphi\left(\lambda\left(1-\frac{j-1}{n}\right)\right) \\
&= -\lambda\sum_{j=1}^{n}\left(-\frac{1}{\lambda}\log\varphi\left(\lambda\left(1-\frac{j-1}{n}\right)\right)\right) \\
&= -\lambda\sum_{j=1}^{n}\psi\left(\frac{j-1}{n}\right) = -\lambda\sum_{j=0}^{n-1}\psi_n(j),
\end{aligned} \tag{2.18}$$

where $\psi_n(z) := \psi\left(\frac{z}{n}\right)$.
Applying the Euler-Maclaurin summation formula (see Gel'fond [17], p.281, formula (66)), we obtain

$$\begin{aligned}
\sum_{j=0}^{n-1}\psi_n(j) &= \int_0^n \psi_n(t)dt + B_1\left(\psi_n(n) - \psi_n(0)\right) \\
&\quad - \frac{1}{2}\int_0^1 \left(B_2(t) - B_2\right)\sum_{j=0}^{n-1}\psi_n''(j+1-t)dt,
\end{aligned} \tag{2.19}$$

where B_k and $B_k(t)$ are Bernoulli numbers and Bernoulli polynomials respectively.
Noting that $\psi_n(n) = \psi(1) = 0 = \psi(0) = \psi_n(0)$, we conclude that the first correction term in (2.19) disappears. Furthermore, by the definition of ψ_n,

$$\int_0^n \psi_n(t)dt = \int_0^n \psi\left(\frac{t}{n}\right)dt = n\int_0^1 \psi(t)dt = nI.$$

Consequently, the equality (2.19) reduces to

$$\sum_{j=0}^{n-1} \psi_n(j) = nI - \frac{1}{2} \int_0^1 (B_2(t) - B_2) \sum_{j=0}^{n-1} \psi_n''(j+1-t)dt. \tag{2.20}$$

Since $\varphi(z)$, $\varphi'(z)$ and $\varphi''(z)$ are bounded on the interval $[0, \lambda]$, we get

$$\sup_{z\in[0,n]} |\psi_n''(z)| = \frac{1}{n^2} \sup_{z\in[0,1]} |\psi''(z)| = \frac{\lambda}{n^2} \sup_{z\in[0,\lambda]} \left| \frac{\varphi''(z)\varphi(z) - (\varphi'(z))^2}{\varphi^2(z)} \right| = \frac{c}{n^2}.$$

Therefore,

$$\left| \int_0^1 (B_2(t) - B_2) \sum_{i=0}^{n-1} \psi_n''(j+1-t)dt \right| \le \frac{c}{n} \int_0^1 |B_2(t) - B_2| dt = O\left(\frac{1}{n}\right).$$

Combining this estimate with (2.20), we obtain

$$\sum_{j=0}^{n-1} \psi_n(j) = nI + O\left(\frac{1}{n}\right). \tag{2.21}$$

Taking into account (2.18) we conclude that (2.17) is valid. Thus, the proof of the lemma is complete. □

Using (2.16) we can derive the upper bound (2.9) for $\mathbf{P}(A_\tau > x)$. Obviously,

$$\mathbf{P}(A_\tau \ge x) = \sum_{n=0}^{\infty} \mathbf{P}(A_n \ge x, \tau = n+1).$$

Using the exponential Chebyschev inequality and recalling that

$$A_n = \sum_{i=1}^{n} (n-j+1)X_j,$$

we obtain

$$\begin{aligned} \mathbf{P}(A_n \ge x, \tau = n+1) &\le \mathbf{P}(A_n \ge x) \\ &\le e^{-\frac{\lambda}{n}x} \mathbf{E} e^{\frac{\lambda}{n}A_n} = e^{-\frac{\lambda}{n}x} \prod_{i=1}^{n} \mathbf{E} e^{\lambda \frac{n-j+1}{n} X_j} \\ &= e^{-\frac{\lambda}{n}x} \prod_{i=1}^{n} \varphi\left(\lambda \frac{n-j+1}{n}\right). \end{aligned}$$

Applying Lemma 4, we get

$$\begin{aligned} \mathbf{P}(A_n \ge x, \tau = n+1) &\le \exp\left\{-\lambda \frac{x}{n} - \lambda I n + O(n^{-1})\right\} \\ &\le C \exp\left\{-\lambda \frac{x}{n} - \lambda I n\right\}. \end{aligned} \tag{2.22}$$

Consequently,

$$\mathbf{P}(A_\tau \ge x) = \sum_{n=1}^{\infty} \mathbf{P}(A_n \ge x, \tau = n+1) \le C \sum_{n=1}^{\infty} \exp\left\{-\lambda \frac{x}{n} - \lambda I n\right\}. \tag{2.23}$$

With formula (25) on page 146 of Batemann [3] we have

$$\begin{aligned}\sum_{n=1}^{\infty} \exp\left\{-\lambda \frac{x}{n} - \lambda I n\right\} &\le \int_0^{\infty} \exp\left\{-\lambda \frac{x}{y} - \lambda I(y+1)\right\} dy \\ &= e^{-\lambda I} \sqrt{\frac{4x}{I}} K_1(2\lambda\sqrt{Ix}).\end{aligned}$$

Now using the asymptotics for the modified Bessel function

$$K_1(z) = \sqrt{\frac{\pi}{2z}} e^{-z},$$

we obtain

$$\sum_{n=1}^{\infty} \exp\left\{-\lambda \frac{x}{n} - \lambda I n\right\} \le C x^{1/4} \exp\{-2\lambda\sqrt{Ix}\}. \tag{2.24}$$

From this bound and (2.23) we obtain (2.9).

2.5 Local limit theorems

We start by proving a standard Gnedenko local limit theorem for the two-dimensional vector $(S_{[nt]}, A_{[nt]})$ under the measure $\widehat{\mathbb{P}}$. The following statement is a one-dimensional case of Theorem 4.2 in Dobrushin and Hryniv [11] and we give its proof for completeness reasons only.

Proposition 2.4. *Assume that the conditions of Theorem 2.1 are valid. Then, for every* $t \in (0,1]$,

$$\sup_{x,y} \left| n^2 \widehat{\mathbb{P}}(S_{[nt]} = x, A_{[nt]} = y) - f_t\left(\frac{x - n\psi(t)}{\sqrt{n}}, \frac{y - n^2 \int_0^t \psi(s)ds}{n^{3/2}}\right)\right| \longrightarrow 0,$$

where f_t *is defined in* (2.14) *and* (2.15).

Proof.
Consider centered random variables

$$X_j^0 := X_j - \widehat{\mathbf{E}} X_j$$

and their characteristic functions

$$\varphi_j(v) := \widehat{\mathbf{E}} e^{ivX_j^0}, \quad 1 \le j \le n.$$

By the inversion formula,

$$\begin{aligned}
&\widehat{\mathbb{P}}(S_{[nt]} = x, A_{[nt]} = y) \\
&\quad = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-iv_1 x - iv_2 y} \widehat{\mathbf{E}} e^{iv_1 S_{[nt]} + iv_2 A_{[nt]}} dv_1 dv_2 \\
&\quad = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-iv_1 n^{1/2} x_0 - iv_2 n^{3/2} y_0} \prod_{j=1}^{n} \varphi_j (v_1 + (n-j+1) v_2) dv_1 dv_2,
\end{aligned}$$

where

$$x_0 := \frac{x - \widehat{\mathbf{E}} S_{[nt]}}{n^{1/2}} \quad \text{and} \quad y_0 := \frac{y - \widehat{\mathbf{E}} A_{[nt]}}{n^{3/2}}.$$

Using the change of variables $v_1 \to \sqrt{n} v_1$, $v_2 \to n^{3/2} v_2$, we get

$$\begin{aligned}
&n^2 \widehat{\mathbb{P}}(S_{[nt]} = x, A_{[nt]} = y) \\
&\quad = \int_{-\pi n^{1/2}}^{\pi n^{1/2}} \int_{-\pi n^{3/2}}^{\pi n^{3/2}} e^{-iv_1 x_0 - iv_2 y_0} \prod_{j=1}^{n} \varphi_j \left(\frac{v_1}{n^{1/2}} + \frac{(n-j+1) v_2}{n^{3/2}} \right) dv_1 dv_2. \qquad (2.25)
\end{aligned}$$

By the inversion formula for Furier transforms,

$$\begin{aligned}
&f_t \left(\frac{x - \widehat{\mathbf{E}} S_{[nt]}}{n^{1/2}}, \frac{x - \widehat{\mathbf{E}} A_{[nt]}}{n^{3/2}} \right) \\
&\quad = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iv_1 x_0 - iv_2 y_0} e^{-(v_1, v_2) \Sigma_t (v_1, v_2)^T} dv_1 dv_2. \qquad (2.26)
\end{aligned}$$

Define

$$\begin{aligned}
R_2 &= \{(v_1, v_2) : v_1 \in [-\varepsilon n^{1/2}, \varepsilon n^{1/2}], v_2 \in [-\varepsilon n^{3/2}, \varepsilon n^{3/2}]\}; \\
R_3 &= \{(v_1, v_2) : v_1 \in [-\pi n^{1/2}, \pi n^{1/2}], v_2 \in [-\pi n^{3/2}, \pi n^{3/2}]\}.
\end{aligned}$$

Combining (2.25) and (2.26), we conclude that

$$\begin{aligned}
&\sup_{x,y} \left| n^2 \widehat{\mathbb{P}}(S_{[nt]} = x, A_{[nt]} = y) - f_t \left(\frac{x - \mathbf{E} S_{[nt]}}{\sqrt{n}}, \frac{y - \mathbf{E} A_{[nt]}}{n^{3/2}} \right) \right| \\
&\quad \le I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{(2\pi)^2} \int_{-A}^{A} \int_{-B}^{B} \left| \prod_{j=1}^{n} \varphi_j \left(\frac{v_1}{n^{1/2}} + \frac{(n-j+1) v_2}{n^{3/2}} \right) - e^{(v_1, v_2) \Sigma_t (v_1, v_2)^T} \right| dv_1 dv_2, \\
I_2 &= \frac{1}{(2\pi)^2} \iint_{R_2 \setminus [-A, A] \times [-B, B]} \prod_{j=1}^{n} \left| \varphi_j \left(\frac{v_1}{n^{1/2}} + \frac{(n-j+1) v_2}{n^{3/2}} \right) \right| dv_1 dv_2, \\
I_3 &= \frac{1}{(2\pi)^2} \iint_{R_3 \setminus R_2} \prod_{j=1}^{n} \left| \varphi_j \left(\frac{v_1}{n^{1/2}} + \frac{(n-j+1) v_2}{n^{3/2}} \right) \right| dv_1 dv_2, \\
I_4 &= \frac{1}{(2\pi)^2} \int_{|v_1| > A} \int_{|v_2| > B} e^{-(v_1, v_2) \Sigma_t (v_1, v_2)^T} dv_1 dv_2.
\end{aligned}$$

Choosing A and B large enough, we can make the integral I_4 as small as we please. Furthermore, the weak convergence

$$\left(\frac{S_{[nt]}-\widehat{\mathbf{E}}S_{[nt]}}{\sqrt{n}},\frac{A_{[nt]}-\widehat{\mathbf{E}}A_{[nt]}}{n^{3/2}}\right)\Rightarrow\left(\xi(t),\int_0^t\xi(s)ds\right)$$

implies that, uniformly on every compact $[-A,A]\times[-B,B]$,

$$\left|\prod_{j=1}^n\varphi_j\left(\frac{v_1}{n^{1/2}}+\frac{(n-j+1)v_2}{n^{3/2}}\right)-e^{(v_1,v_2)\Sigma_t(v_1,v_2)^T}\right|\to 0.$$

Consequently, I_1 converges to zero.
It is clear that the random variables X_j^2 are uniformly integrable with respect to the measure $\widehat{\mathbb{P}}$. Therefore, for every ε small enough,

$$|\varphi_j(v)|\le e^{-\sigma^2(u_{n,j})v^2/4},\quad |v|\le 2\varepsilon,\ 1\le j\le n.$$

Consequently, there is an existance of constants $c>0$ and C such that

$$\begin{aligned}\prod_{j=1}^n\left|\varphi_j\left(\frac{v_1}{n^{1/2}}+\frac{v_2}{n^{3/2}}\right)\right| &\le \exp\left\{-\sum_{j=1}^n\frac{\sigma^2(u_{n,j})}{4}\left(\frac{v_1}{\sqrt{n}}+\frac{(n-j+1)v_2}{n^{3/2}}\right)^2\right\}\\ &\le C\exp\left\{-c(v_1,v_2)\Sigma_t(v_1,v_2)^T\right\}\end{aligned}$$

on the set $|v_1|\le\varepsilon n^{1/2}$, $|v_2|\le\varepsilon n^{1/2}$. Therefore, I_2 can be made as small as we please by choosing A and B large enough.

It remains to bound I_3. Since the distributions of random variables X_j are aperiodic, $|\widehat{\mathbf{E}}[e^{ivX_j}]|=1$ if and only if $v=2\pi m$. Furthermore, recalling that the distributions of X_j are obtained via the exponentail change of measure of the same distribution and that the parameters of these changes are taken from the bounded interval, we conclude that for every $\delta>0$ there exists $c_\delta>0$ such that

$$\max_{1\le j\le n}|\varphi_j(v)|\le e^{-c_\delta}\quad\text{for all } v \text{ such that } |v-2\pi m|>\delta \text{ for all } m\in\mathbb{Z}. \tag{2.27}$$

For all v_1,v_2 from the integration region in I_3, we have the following property. At least $\frac{n}{2}$ elements of the sequence $\left\{\frac{v_1}{\sqrt{n}}+\frac{(n-j+1)v_2}{n^{3/2}}\right\}_{j=1}^n$ are separated from the set $\{2\pi m, m\in\mathbb{Z}\}$. From this fact and (2.27) we infer that there exists δ_0 such that

$$\prod_{j=1}^n\left|\varphi_j\left(\frac{v_1}{n^{1/2}}+\frac{v_2}{n^{3/2}}\right)\right|\le e^{-c_{\delta_0}n/2}.$$

Consequently, I_3 converges to zero as $n\to\infty$. Thus, the proof is complete. □

Proposition 2.5. *Assume that the conditions of Theorem 2.1 are valid. Then there exists a positive, increasing function $q(a)$ such that, for every $t\in(0,1)$ and every $a\ge 0$,*

$$\begin{aligned}\sup_{x,y}\Bigg|n^2\widehat{\mathbb{P}}&\left(S_{[nt]}=x, A_{[nt]}=y, \min_{k\le[nt]}S_k>-a\right)\\ &-q(a)f_t\left(\frac{x-n\psi(t)}{\sqrt{n}},\frac{y-n^2\int_0^t\psi(s)ds}{n^{3/2}}\right)\Bigg|\to 0.\end{aligned} \tag{2.28}$$

Proof.
Set $m = [\log^2 n]$. Then, by the Markov property at time m,

$$\widehat{\mathbb{P}}(S_{[nt]} = x, A_{[nt]} = y, \min_{k\le n} S_k > -a) \\ = \sum_{x',y'>0} \widehat{\mathbb{P}}(S_m = x', A_m = y', \min_{k\le m} S_k > -a) Q(x'y'; x, y), \tag{2.29}$$

where

$$Q(x'y'; x, y) \\ = \widehat{\mathbb{P}}\left(S^{(m)}_{[nt]-m} = x - x', A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x', \min_{k\le[nt]-m} S^{(m)}_k > -x' - a\right)$$

and

$$S^{(m)}_k = X_{m+1} + \cdots X_{m+k} \quad \text{and} \quad A^{(m)}_k = S^{(m)}_1 + S^{(m)}_2 + \ldots + S^{(m)}_k.$$

By Proposition 2.4,

$$Q(x'y'; x, y) \le \widehat{\mathbb{P}}\left(S^{(m)}_{[nt]-m} = x - x', A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x'\right) \le \frac{c_t}{n^2}. \tag{2.30}$$

It follows from the definition of $\widehat{\mathbb{P}}$ that the second moments of X_j are uniformly bounded. Applying the Chebyshev inequality, we then obtain

$$\widehat{\mathbb{P}}(|S_m - \widehat{\mathbf{E}}S_m| \ge \log^{3/2} n) = o(1) \tag{2.31}$$

and

$$\widehat{\mathbb{P}}(|A_m - \widehat{\mathbf{E}}A_m| \ge \log^{5/2} n) = o(1). \tag{2.32}$$

Define

$$D := \left\{(x', y') : |x' - \widehat{\mathbf{E}}S_m| \le \log^{3/2} n, |y' - \widehat{\mathbf{E}}A_m| \le \log^{5/2} n\right\}.$$

Combining (2.30), (2.31) and (2.32), we conclude that, uniformly in $x, y > 0$,

$$\lim_{n\to\infty} n^2 \sum_{D^c} \widehat{\mathbb{P}}(S_m = x', A_m = y', \min_{k\le m} S_k > -a) Q(x'y'; x, y) = 0. \tag{2.33}$$

We turn now to the asymptotic behaviour of $Q(x'y'; x, y)$ for (x', y') belonging to the set D. Obviously,

$$Q(x'y'; x, y) = \widehat{\mathbb{P}}\left(S^{(m)}_{[nt]-m} = x - x', A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x'\right) \tag{2.34} \\ - \widehat{\mathbb{P}}\left(S^{(m)}_{[nt]-m} = x - x', A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x', \min_{k\le[nt]-m} S^{(m)}_k \le -x' - a\right).$$

We can apply Proposition 2.4 to the first probabilty term on the right hand side of (2.34). As a result, uniformly in $x, x', y, y' > 0$,

$$n^2 \widehat{\mathbb{P}}\left(S^{(m)}_{[nt]-m} = x - x', A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x'\right) \\ - f_t\left(\frac{x - x' - n\psi(t)}{\sqrt{n}}, \frac{y - y' - n^2 \int_0^t \psi(s)ds}{n^{3/2}}\right) \to 0. \tag{2.35}$$

Furthermore, it follows easily from the definition of the measure $\widehat{\mathbb{P}}$ that $\widehat{\mathbf{E}}X_j \sim \frac{\varphi'(\lambda)}{\varphi(\lambda)}$ for each $j \le m$. Therefore, $\widehat{\mathbf{E}}S_m \sim \frac{\varphi'(\lambda)}{\varphi(\lambda)}\log^2 n$ and $\widehat{\mathbf{E}}A_m \sim \frac{\varphi'(\lambda)}{2\varphi(\lambda)}\log^4 n$. From these relations we infer that

$$f_t\left(\frac{x-x'-n\psi(t)}{\sqrt{n}}, \frac{y-y'-n^2\int_0^t\psi(s)ds}{n^{3/2}}\right) \\ - f_t\left(\frac{x-n\psi(t)}{\sqrt{n}}, \frac{y-n^2\int_0^t\psi(s)ds}{n^{3/2}}\right) \to 0$$

uniformly in $x, y > 0$ and $(x', y') \in D$. Combining this with (2.35), we conclude that

$$n^2\widehat{\mathbb{P}}\left(S^{(m)}_{[nt]-m} = x - x', A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x'\right) \\ - f_t\left(\frac{x-n\psi(t)}{\sqrt{n}}, \frac{y-n^2\int_0^t\psi(s)ds}{n^{3/2}}\right) \to 0 \tag{2.36}$$

uniformly in $x, y > 0$ and $(x', y') \in D$.
Moreover, for every $(x', y') \in D$ and all n sufficiently large we have

$$\widehat{\mathbb{P}}\left(S^{(m)}_{[nt]-m} = x - x', A^{(m)}_{[nt]-m} = y - y' - ([nt]-m)x', \min_{k\le[nt]-m} S^{(m)}_k \le -x'\right) \\ \le \widehat{\mathbb{P}}\left(\min_{k\le[nt]-m} S^{(m)}_k \le -x'\right) \le \widehat{\mathbb{P}}\left(\min_{k\le[nt]-m} S^{(m)}_k \le -\log^{3/2} n\right).$$

By the exponential Chebyshev inequality,

$$\widehat{\mathbb{P}}(S_k \le -\log^{3/2} n) = \widehat{\mathbb{P}}(-S_k \ge \log^{3/2} n) \le e^{-\lambda h \log^{3/2} n}\widehat{\mathbf{E}}e^{-\lambda h S_k}. \tag{2.37}$$

Furthermore, it follows from the definition of $\widehat{\mathbb{P}}$ that

$$\widehat{\mathbf{E}}e^{-\lambda h S_k} = \prod_{j=1}^k \widehat{\mathbf{E}}e^{-\lambda h X_j} = \prod_{j=1}^k \frac{\varphi(u_{n,j}-\lambda h)}{\varphi(u_{n,j})} \\ = \exp\left\{-\lambda\sum_{j=1}^k \psi\left(\frac{j-1}{n}+h\right) + \lambda\sum_{j=1}^k \psi\left(\frac{j-1}{n}\right)\right\}. \tag{2.38}$$

Using here the Euler-Maclaurin summation formula (2.19), we infer that

$$\sum_{j=1}^k \psi\left(\frac{j-1}{n}\right) - \sum_{j=1}^k \psi\left(\frac{j-1}{k}+h\right) \le c + n\left(\int_0^{k/n}\psi(u)du - \int_h^{h+k/n}\psi(u)du\right).$$

If $h < 1-t$ then the function $s \mapsto \int_0^s \psi(u)du - \int_h^{h+s}\psi(u)du$ achieves its maximum either at zero or at t. Therefore,

$$\max_{s\in[0,t]}\left(\int_0^s \psi(u)du - \int_h^{s+h}\psi(u)du\right) = \left(\int_0^t \psi(u)du - \int_h^{t+h}\psi(u)du\right)^+.$$

If h is so small that $\psi(h) < \psi(t+h)$, then

$$\int_0^t \psi(u)du - \int_h^{t+h} \psi(u)du < 0$$

and consequently,

$$\max_{k\le nt}\left(\sum_{j=1}^k \psi\left(\frac{j-1}{n}\right) - \sum_{j=1}^k \psi\left(\frac{j-1}{n}+h\right)\right) \le c.$$

Plugging this into (2.38), we obtain

$$\max_{k\le nt} \widehat{\mathbf{E}} e^{-\lambda h S_k} \le e^c.$$

Combining this estimate and (2.37) we finally get

$$\begin{aligned}\widehat{\mathbb{P}}\left(\min_{k\le nt} S_k \le -\log^{3/2} n\right) &\le \sum_{j=1}^{nt} \widehat{\mathbb{P}}\left(S_k < -\log^{3/2} n\right)\\ &\le nte^c e^{-\lambda h \log^{3/2} n} = o\left(\frac{1}{n^2}\right).\end{aligned}$$

So we get, uniformly in $x, y > 0$ and $(x', y') \in D$,

$$n^2 Q(x', y'; x, y) - \frac{1}{n^2} f_t\left(\frac{x - n\psi(t)}{\sqrt{n}}, \frac{y - n^2\int_0^t \psi(s)ds}{n^{3/2}}\right) \to 0 \tag{2.39}$$

Combining (2.39), (2.31) and (2.32), we conclude that

$$\begin{aligned}&n^2 \sum_D \widehat{\mathbb{P}}(S_m = x', A_m = y', \min_{k\le m} S_k > -a) Q(x'y'; x, y)\\ &= f_t\left(\frac{x - n\psi(t)}{\sqrt{n}}, \frac{y - n^2\int_0^t \psi(s)ds}{n^{3/2}}\right) \sum_D \widehat{\mathbb{P}}(S_m = x', A_m = y', \min_{k\le m} S_k > -a) + o(1)\\ &= f_t\left(\frac{x - n\psi(t)}{\sqrt{n}}, \frac{y - n^2\int_0^t \psi(s)ds}{n^{3/2}}\right) \widehat{\mathbb{P}}\left(\min_{k\le m} S_k > -a\right) + o(1). \end{aligned}\tag{2.40}$$

For every fixed $m_0 \ge 1$ and all $m \ge m_0$ we have

$$\widehat{\mathbb{P}}\left(\min_{k\le m} S_k > -a\right) \le \widehat{\mathbb{P}}\left(\min_{k\le m_0} S_k > -a\right)$$

and

$$\widehat{\mathbb{P}}\left(\min_{k\le m} S_k > -a\right) \ge \widehat{\mathbb{P}}\left(\min_{k\le m_0} S_k > -a\right) - \widehat{\mathbb{P}}\left(\min_{m_0<k\le m} S_k > a\right).$$

For the second probability term on the right hand side we have

$$\begin{aligned}
&\widehat{\mathbb{P}}\left(\min_{m_0<k\le m} S_k > a\right)\\
&\qquad \le \widehat{\mathbb{P}}\left(S_{m_0} < m_0^{2/3}\right) + \widehat{\mathbb{P}}\left(\min_{k\le m-m_0} S_k < -m_0^{2/3}\right)\\
&\qquad \le \widehat{\mathbb{P}}\left(S_{m_0} < m_0^{2/3}\right) + \sum_{k=1}^{m-m_0} \widehat{\mathbb{P}}\left(S_k < -m_0^{2/3}\right).
\end{aligned}$$

Using the exponential Chebyshev inequality once again, one can easily infer that there exists $f(x) \to 0$, $x \to \infty$ such that, for all $n \ge 1$,

$$\widehat{\mathbb{P}}\left(\min_{m_0<k\le m} S_k < -a\right) \le f(m_0).$$

Consequently,

$$\widehat{\mathbb{P}}\left(\min_{k\le m_0} S_k > -a\right) - f(m_0) \le \widehat{\mathbb{P}}\left(\min_{k\le m} S_k > -a\right) \le \widehat{\mathbb{P}}\left(\min_{k\le m_0} S_k > -a\right).$$

For every $j \le m_0$ the distribution of X_j converges, as $n \to \infty$, to the distribution of X_1 under $\widehat{\mathbb{P}}$. (Here one has to notice that this distribution does not depend on n.) Let U_k denote a random walk with i.i.d. increments, which are distributed according to the limiting distribution of X_j. Then

$$\lim_{n\to\infty} \widehat{\mathbb{P}}\left(\min_{k\le m_0} S_k > -a\right) = \mathbf{P}\left(\min_{k\le m_0} U_k > -a\right).$$

Letting now $m_0 \to \infty$, we finally get

$$\lim_{n\to\infty} \widehat{\mathbb{P}}\left(\min_{k\le m} S_k > -a\right) = \mathbb{P}\left(\min_{k\ge 1} U_k \ge -a\right) =: q(a).$$

The positivity of the function q follows from the fact that the increments of U_k have a positive mean. Applying the previous relation to (2.40) and taking into account (2.33), we obtain the desired asymptotics. □

In order to prove local limit theorems for (S_n, A_n) conditioned on $\{\tau > n, S_n = x\}$ with fixed x we are going to consider the path $\{S_{[n/2]}, S_{[n/2]+1}, \dots, S_n\}$ in the reversed time. More precisely, we shall consider random variables

$$\widehat{X}_k = -X_{n-k+1} \quad \text{and} \quad \widehat{S}_k = \widehat{X}_1 + \widehat{X}_2 + \ldots + \widehat{X}_k, \quad k = 1, 2, \dots, n.$$

Proposition 2.6. *Assume that the conditions of Theorem 2.1 are valid. Then there exists a positive increasing $\widehat{q}$ such that, for every $t \in (0,1)$,*

$$\begin{aligned}
&n^2 \widehat{\mathbb{P}}\left(\widehat{S}_{[nt]} = x, \widehat{A}_{[nt]-1} = y, \min_{k\le[nt]} \widehat{S}_k > -a\right)\\
&\qquad - \widehat{q}(a)\widehat{f}_t\left(\frac{x - n\psi(1-t)}{\sqrt{n}}, \frac{y - n^2\int_{1-t}^1 \psi(s)ds}{n^{3/2}}\right) \to 0 \qquad (2.41)
\end{aligned}$$

uniformly in $x, y > 0$. The function $\widehat{f}_t$ is the density function of the normal distribution with zero mean and the covariance matrix

$$\widehat{\Sigma}_t = \begin{pmatrix} \int_{1-t}^1 \sigma^2(u)du & \int_{1-t}^1 \sigma^2(u)(t-1+u)du \\ \int_{1-t}^1 \sigma^2(u)(t-1+u)du & \int_{1-t}^1 \sigma^2(u)(t-1+u)^2du \end{pmatrix}.$$

The proof of this proposition repeats that of Propositions 2.4 and 2.5 and we omit it. We now state a local limit theorem for a bridge of S_n conditioned to stay positive. This result is the most important ingredient in our approach to the proof of Theorem 2.1.

Proposition 2.7. *Assume that the conditions of Theorem 2.1 are valid. Then, for every fixed x,*

$$n^2\widehat{\mathbb{P}}\left(A_n = y, S_n = x, \tau > n\right) - q(0)\widehat{q}(x)f_1\left(0, \frac{y-n^2I}{n^{3/2}}\right) \to 0. \tag{2.42}$$

Proof.
It is immediate from the definition of $\widehat{S}_k$ that $S_k = S_n - \sum_{j=k+1}^n X_j = S_n + \widehat{S}_{n-k}$. Therefore, for $\ell(n) = [nt]$ with some fixed $t \in (0,1)$ we have

$$\begin{aligned} \{A_n = y, S_n = x\} &= \left\{A_{l(n)} + \sum_{k=l(n)+1}^n S_k = y, S_{l(n)} - \widehat{S}_{n-l(n)} = x\right\} \\ &= \left\{A_{l(n)} + (n-l(n))x + \sum_{l(n)+1}^n \widehat{S}_{n-k} = y, S_{l(n)} - \widehat{S}_{n-l(n)} = x\right\} \\ &= \left\{A_{l(n)} + \widehat{A}_{n-l(n)-1} = y - (n-l(n))x, S_{l(n)} - \widehat{S}_{n-l(n)} = x\right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\widehat{\mathbb{P}}\{A_n = y, S_n = x, \tau > n\} \\ &\quad= \widehat{\mathbb{P}}\left\{A_{l(n)} + \widehat{A}_{n-l(n)-1} = y - (n-l(n))x, S_{l(n)} - \widehat{S}_{n-l(n)} = x, \tau > n\right\} \\ &\quad= \sum_{x',y'} \widehat{\mathbb{P}}(A_{l(n)} = y', S_{l(n)} = x', \tau > l(n))\widehat{Q}(x', y'; x, y), \end{aligned}$$

where

$$\begin{aligned} &\widehat{Q}(x', y'; x, y) \\ &\quad:= \widehat{\mathbb{P}}\left(\widehat{A}_{n-l(n)-1} = y - y' - (n-l(n))x, \widehat{S}_{n-l(n)} = x' - x, \min_{k \le n-\ell(n)} \widehat{S}_k > -x\right). \end{aligned}$$

Combining Propositions 2.5 and 2.6, we conclude that, for every fixed x,

$$n^2\widehat{\mathbb{P}}\left(A_n = y, S_n = x, \tau > n\right) \sim q(0)\widehat{q}(x)n^2\Sigma_n(y),$$

where

$$\begin{aligned} \Sigma_n(y) := \sum_{x',y'} f_t &\left(\frac{x' - n\psi(1/2)}{\sqrt{n}}, \frac{y' - n^2\int_0^{1/2}\psi(s)ds}{n^{3/2}}\right) \\ &\times \widehat{f}_{1-t}\left(\frac{x' - n\psi(1/2)}{\sqrt{n}}, \frac{y - y' - n^2\int_{1/2}^1 \psi(s)ds}{n^{3/2}}\right). \end{aligned}$$

It is immediate from the continuity and boundedness of functions f_t and $\widehat{f}_{1-t}$ that

$$n^2\Sigma_n(y) \sim \int_{\mathbb{R}^2} f_t(u,v)\widehat{f}_{1-t}\left(u, \frac{y-n^2I}{n^{3/2}} - v\right) dudv, \quad n \to \infty$$

and consequently,

$$n^2\widehat{\mathbb{P}}\{A_n = y, S_n = x, \tau > n\} \sim q(0)\widehat{q}(x) \int_{\mathbb{R}^2} f_t(u,v)\widehat{f}_{1-t}\left(u, \frac{y-n^2I}{n^{3/2}} - v\right) dudv.$$

Since the left hand side does not depend on t, we infer that the integral on the right hand side does not depend on t as well. Letting $t \to 1$ and using continuity of f_t, we infer that

$$\int_{\mathbb{R}^2} f_t(u,v)\widehat{f}_{1-t}(u, z-v)\, dudv = f_1(0,z).$$

This completes the proof of the proposition. □

2.6 Proofs of tail asymptotics

2.6.1 Proof of Theorem 2.1

Using (2.12), we obtain

$$\begin{aligned}
\mathbb{P}(A_n = x, &\tau = n+1) \\
&= \sum_{y=1}^{\infty} \mathbb{P}(A_n = x, S_n = y, \tau = n+1) \\
&= \sum_{y=1}^{\infty} \mathbb{P}(A_n = x, S_n = y, \tau > n)\mathbb{P}(X_{n+1} \le -y) \\
&= e^{-\lambda x/n}\prod_{j=1}^{n}\varphi(u_{n,j})\sum_{y=1}^{\infty}\widehat{\mathbb{P}}(A_n = x, S_n = y, \tau > n)\mathbb{P}(X_{n+1} \le -y).
\end{aligned}$$

It follows from Proposition 2.6 that, for every fixed M,

$$\begin{aligned}
\sum_{y=1}^{M} &\widehat{\mathbb{P}}(A_n = x, S_n = y, \tau > n)\mathbb{P}(X_{n+1} \le -y) \\
&= \frac{q(0)}{n^2} h\left(\frac{x-n^2I}{n^{3/2}}\right)\sum_{y=1}^{M}\widehat{q}(y)\mathbb{P}(X_1 \le -y) + o\left(\frac{1}{n^2}\right). \qquad (2.43)
\end{aligned}$$

Furthermore, applying Proposition 2.4, we have

$$\begin{aligned}
\sum_{y=M+1}^{\infty} &\widehat{\mathbb{P}}(A_n = x, S_n = y, \tau > n)\mathbb{P}(X_{n+1} \le -y) \\
&\le \sum_{y=M+1}^{\infty} \widehat{\mathbb{P}}(A_n = x, S_n = y)\mathbb{P}(X_{n+1} \le -y) \le \frac{c}{n^2}\sum_{M+1}^{\infty}\mathbb{P}(X_1 \le -y).
\end{aligned}$$

Consequently, uniformly in n,

$$\lim_{M\to\infty} n^2 \sum_{y=M+1}^{\infty} \widehat{\mathbb{P}}(A_n = x, S_n = y, \tau > n)\mathbb{P}(X_{n+1} \le -y) = 0. \tag{2.44}$$

Combining (2.43) and (2.44), we conclude that

$$\begin{aligned}&\sum_{y=1}^{\infty}\widehat{\mathbb{P}}(A_n = x, S_n = y; \tau > n)\mathbb{P}(X_{n+1} \le -y)\\ &\qquad = \frac{1}{n^2} h\left(\frac{x-n^2 I}{n^{3/2}}\right)\sum_{y=1}^{\infty}\widehat{q}(y)\mathbb{P}(X_1 \le -y) + o\left(\frac{1}{n^2}\right).\end{aligned}$$

According to Lemma 2.3,

$$\prod_{j=1}^{n}\varphi(u_{n,j}) = \exp\{-\lambda I n\}\left(1+O(n^{-1})\right).$$

Therefore,

$$\mathbb{P}(A_n = x, \tau = n+1) = \frac{Q+o(1)}{n^2}\exp\left\{-\frac{\lambda x}{n} - \lambda n I\right\} h\left(\frac{x-n^2 I}{n^{3/2}}\right), \tag{2.45}$$

where

$$Q := q(0)\sum_{y=1}^{\infty}\widehat{q}(y)\mathbb{P}(X_1 \le -y).$$

In particular, there exists a constant C, such that

$$\mathbb{P}(A_n, \tau = n+1) \le \frac{C}{n^2}\exp\left\{-\frac{\lambda x}{n} - \lambda n I\right\}. \tag{2.46}$$

Define: $n_- = \lfloor t_0 \rfloor = \max\{n \in \mathbb{N} : n \le t_0\}$ and $n_+ = n_- + 1$, where $t_0 = \sqrt{\frac{x}{I}}$. Changing the summation index and splitting the series into two parts, we get

$$\begin{aligned}\sum_{n=n_+}^{\infty}\mathbb{P}(A_n = x, \tau = n+1) &= \sum_{k=0}^{\infty}\mathbb{P}\left(A_{n_++k} = x, \tau = n_+ + k + 1\right)\\ &= \sum_{k\le M n_+^{1/2}}\mathbb{P}(A_{n_++k} = x, \tau = n_+ + k + 1)\\ &\quad + \sum_{k > M n_+^{1/2}}\mathbb{P}(A_{n_++k} = x, \tau = n_+ + k + 1).\end{aligned} \tag{2.47}$$

Applying (2.45) to the summands in the first sum, we get

$$\begin{aligned}&\sum_{k\le M n_+^{1/2}}\mathbb{P}(A_{n_++k} = x, \tau = n_+ + k + 1)\\ &= \frac{Q}{n_+^2}\sum_{k\le M n_+^{1/2}}\exp\left\{-\frac{\lambda x}{n_+ + k} - \lambda I(n_+ + k)\right\} h\left(\frac{x-(n_+ + k)^2 I}{(n_+ + k)^{3/2}}\right) + o\left(n_+^{-3/4}\right).\end{aligned}$$

Since $x - n_+^2 I + k^2 I = o(n^{3/2})$ uniformly in $k \le M n_+^{1/2}$,

$$h\left(\frac{x-(n_++k)^2 I}{(n_++k)^{3/2}}\right) \sim h\left(-\frac{2Ik}{n_+^{1/2}}\right). \tag{2.48}$$

Furthermore,

$$\begin{aligned}\frac{\lambda x}{n_++k} + \lambda I(n_++k) &= \frac{\lambda x}{n_+}\left(1-\frac{k}{n_+}+\frac{k^2}{n_+^2}+O\left(\frac{k^3}{n_+^3}\right)\right)+\lambda I n_+ + \lambda I k \\ &= \left(\frac{\lambda x}{n_+}+\lambda I n_+\right)+\lambda I k - \frac{\lambda x}{n_+^2}k + \frac{\lambda x k^2}{n_+^3} + O\left(\frac{\lambda x}{n_+^{5/2}}\right).\end{aligned}$$

Recalling now that $n_+ = \sqrt{\frac{x}{I}} + \varepsilon_x$ with $\varepsilon_x \in (0,1]$, we have, uniformly in $k \le M n_+^{1/2}$,

$$\begin{aligned}0 \le \lambda I k - \frac{\lambda x}{n_+^2}k &= \left(I - \frac{x}{\frac{x}{I}\left(1+\varepsilon_x\sqrt{\frac{I}{x}}\right)^2}\right)\lambda k \\ &\le 2\lambda k \varepsilon_x \sqrt{\frac{I}{x}} = O\left(x^{-1/4}\right) = O\left(\frac{1}{n_+^{1/2}}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{k \le M n_+^{1/2}} \mathbb{P}(A_{n_++k} = x, \tau = n_+ + k + 1) \\ &= \frac{Q}{n_+^2}\exp\left\{-\frac{\lambda x}{n_+} - \lambda I n_+\right\}\left[\sum_{k \le M n_+^{1/2}} \exp\left\{-\lambda I \frac{k2}{n_+}\right\} h\left(-\frac{2IK}{n_+^{1/2}}\right) + o\left(n_+^{3/2}\right)\right] \\ &= \frac{Q}{n_+^{3/2}}\exp\left\{-\frac{\lambda x}{n_+} - \lambda I n_+\right\}\left[\int_0^M e^{-\lambda I u^2} h(-2Iu)du + o\left(n_+^{-3/2}\right)\right] \\ &= \frac{\widehat{Q}}{x^{3/4}}\exp\left\{-2\lambda\sqrt{Ix}\right\}\left[\int_0^M e^{-\lambda I u^2} h(-2Iu)du + o(1)\right].\end{aligned} \tag{2.49}$$

We split the second sum in (2.47) into two parts: $k \le n_+$ and $k > n_+$. Using (2.46), we get

$$\begin{aligned}&\sum_{k \in (M n_+^{1/2}, n_+]} \mathbb{P}(A_{n_++k} = x, \tau = n_+ + k + 1) \\ &\qquad \le \frac{C}{n_+^2} \sum_{k \in (M n_+^{1/2}, n_+]} \exp\left\{-\frac{\lambda x}{n_+ + k} - \lambda I(n_+ + k)\right\}.\end{aligned}$$

Using now (2.24), we get

$$\begin{aligned}
&\sum_{k\in(Mn_+^{1/2},n_+]} \mathbb{P}(A_{n_++k}=x,\tau=n_++k+1)\\
&\qquad\le \frac{C}{n_+^2}\exp\left\{-\frac{\lambda x}{n_+}-n_+\lambda I\right\}\sum_{k\in(Mn_+^{1/2},n_+]} e^{-\frac{\lambda I}{2}\frac{k^2}{n_+}}\\
&\qquad\le \frac{\widehat{C}}{n_+^{3/2}}\exp\left\{-\frac{\lambda x}{n_+}-n_+\lambda I\right\}\int_M^\infty e^{-\frac{\lambda I u^2}{2}}du. \qquad (2.50)
\end{aligned}$$

For $k>n_+$ we have by (2.22)

$$\begin{aligned}
&\sum_{k>n_+} \mathbb{P}(A_{n_++k}=x,\tau=n_++k+1)\\
&\qquad\le \sum_{k>n_+}\exp\left\{-\frac{\lambda x}{n_++k}-\lambda I(n_++k)\right\}\\
&\qquad\le C\exp\left\{-\frac{\lambda x}{n_+}-\lambda I n_+\right\}\exp\left\{-\frac{n_+\lambda I}{2}\right\}. \qquad (2.51)
\end{aligned}$$

Combining (2.49), (2.50), (2.51) and letting $M\to\infty$, we conclude that, for some $C_+>0$,

$$\sum_{n=n_+}^{\infty}\mathbb{P}(A_n,\tau=n+1)\sim\frac{C_+}{x^{3/4}}\exp\left\{-2\lambda\sqrt{Ix}\right\}.$$

Similar arguments lead to

$$\sum_{n=1}^{n_-}\mathbb{P}(A_n=x,\tau=n+1)\sim\frac{C_-}{x^{3/4}}\exp\left\{-2\lambda\sqrt{Ix}\right\}.$$

Thus the proof of Theorem 2.1 is complete.

2.6.2 Proof of Theorem 2.2

For $k\ge 0$ we have

$$\mathbb{P}(\tau=n_++k+1|A_\tau=x)=\frac{\mathbb{P}(A_{n_++k}=x,\tau=n_++k+1)}{\mathbb{P}(A_\tau=x)}.$$

It follows from (2.45) that

$$\begin{aligned}
\mathbb{P}(A_{n_++k}=x,\tau=n_++k+1)&=\frac{Q}{(n_++k)^2}\exp\left\{-\frac{\lambda x}{n_++k}-\lambda I(n_++k)\right\}\\
&\quad\times\left[f_1\left(0,\frac{x-(n_++k)^2 I}{(n_++k)^{3/2}}\right)+o(1)\right].
\end{aligned}$$

It is immediate from the definition of h that

$$\varepsilon_M:=\max_{u\ge M}f_1(0,u)\to 0\quad\text{as } M\to\infty.$$

Therefore, for all x large enough and all $k \geq M n_+^{1/2}$,

$$\mathbb{P}(\tau = n_+ + k + 1 | A_\tau = x) \leq C\varepsilon_M.$$

For $k < M n_+^{1/2}$ we have from (2.48)

$$\mathbb{P}(A_{n_+ + k} = x, \tau = n_+ + k + 1)$$
$$\sim \frac{Q}{n_+^2} \exp\left\{-\frac{\lambda x}{n_+} - \lambda I n_+\right\} \exp\left\{-\lambda I \frac{k^2}{n_+}\right\} f_1\left(0, -2I \frac{k}{n_+^{1/2}}\right).$$

It follows now from Theorem 2.1 that

$$\mathbb{P}(\tau = n_+ + k + 1 | A_\tau = x) \sim C x^{1/4} \exp\left\{-\lambda I \frac{k^2}{n_+}\right\} f_1\left(0, -2I \frac{k}{n_+^{1/2}}\right).$$

Recalling that

$$f_1(0, z) = c \exp\left\{-\frac{z^2}{2\int_0^1 \sigma^2(u)(1-u)^2 du}\right\}$$

we get the desired asymptotics for $k \geq 0$. The case $k < 0$ can be treated in the same manner.

2.6.3 Proof of (2.1)

Fix some $\varepsilon > 0$. Then

$$\mathbf{P}(A_\tau > x) = \mathbf{P}(A_\tau > x, \tau \leq \varepsilon x^{2/3}) + \sum_{n \geq \varepsilon x^{2/3}} \mathbf{P}(A_\tau > x, \tau = n + 1). \tag{2.52}$$

It is easy to see that $\{A_\tau > x, \tau \leq \varepsilon x^{2/3}\} \subset \{M_\tau > x^{1/3}/\varepsilon\}$. Doney has shown in [12] that $y\mathbf{P}(M_\tau > y) \to c \in (0, \infty)$. Therefore, there exists a constant C such that

$$x^{1/3}\mathbf{P}(A_\tau > x, \tau \leq \varepsilon x^{2/3}) \leq C\varepsilon \quad \text{for all } x > 0. \tag{2.53}$$

By the functional limit theorem for random walk excursions (see Caravenna and Chaumont [5] and Sohier [32]),

$$\mathbf{P}(A_\tau > x | \tau = n + 1) = \overline{G}\left(\frac{x}{\sigma n^{3/2}}\right) + o(1),$$

where

$$\overline{G}(y) := \mathbf{P}\left(\int_0^1 e(t)dt > y\right).$$

Furthermore, according to Theorem 8 in Vatutin and Wachtel [37],

$$\mathbf{P}(\tau = n + 1) \sim \frac{C_0}{n^{3/2}}.$$

Combining these two relations, we obtain

$$\mathbf{P}(A_\tau > x, \tau = n + 1) = \frac{C_0}{n^{3/2}} \overline{G}\left(\frac{x}{\sigma n^{3/2}}\right) + o(n^{-3/2}).$$

and consequently,

$$\sum_{n\geq\varepsilon x^{2/3}} \mathbf{P}(A_\tau > x, \tau = n+1) = C_0 \sum_{n\geq\varepsilon x^{2/3}} n^{-3/2}\overline{G}\left(\frac{x}{\sigma n^{3/2}}\right) + o(x^{-1/3}).$$

Since the sum on the right hand side can be written as a Riemann sum for the function $y^{-3/2}\overline{G}(y^{-3/2})$, we have

$$\begin{aligned}\sum_{n\geq\varepsilon x^{2/3}} \mathbf{P}(A_\tau > x, \tau = n+1) &= \frac{C_0\sigma^{1/3}}{x^{1/3}} \int_{\varepsilon\sigma^{2/3}}^{\infty} y^{-3/2}\overline{G}(y^{-3/2})dy + o(x^{-1/3}) \\ &= \frac{2C_0\sigma^{1/3}}{3x^{1/3}} \int_0^{1/(\varepsilon\sigma)} z^{-2/3}\overline{G}(z)dz + o(x^{-1/3}). \qquad (2.54)\end{aligned}$$

Combining (2.52)–(2.54), we obtain

$$\liminf_{x\to\infty} x^{1/3}\mathbf{P}(A_\tau > x) \geq \frac{2C_0\sigma^{1/3}}{3} \int_0^{1/(\varepsilon\sigma)} z^{-2/3}\overline{G}(z)dz$$

and

$$\limsup_{x\to\infty} x^{1/3}\mathbf{P}(A_\tau > x) \leq \frac{2C_0\sigma^{1/3}}{3} \int_0^{1/(\varepsilon\sigma)} z^{-2/3}\overline{G}(z)dz + C\varepsilon.$$

Letting now $\varepsilon \to 0$, we arrive at the relation

$$\lim_{x\to\infty} x^{1/3}\mathbf{P}(A_\tau > x) = \frac{2C_0\sigma}{3} \int_0^{\infty} z^{-2/3}\overline{G}(z)dz = 2C_0\sigma^{1/3}\mathbf{E}\left(\int_0^1 e(t)dt\right)^{1/3}.$$

2.7 Examples

It is more difficult, than expected to find the constants, that must be calculated to use the Theorem 2.1. At first we are interested in rescaling function

$$\psi(u) := -\frac{1}{\lambda}\log\varphi(\lambda(1-u)), \quad u \in [0,1],$$

where $\varphi(t)$ is the moment generating function of X_1. Then we can also calculate following parameters

$$\theta := 2\lambda\sqrt{I} \quad \text{and} \quad I := \int_0^1 \psi(u)du.$$

Example 2.8. Let Y have a Poisson distribution with Parameter μ and consider $X = Y - n_0$, with $n_0 > \mu$. The condition $n_0 > \mu$ ensures that $\mathbf{E}X < 0$. Then for the moment generating function we obtain:

$$\varphi(t) = e^{-n_0 t}e^{\mu(e^t-1)}.$$

The Cramér condition must be hold, i.e. $\varphi(\lambda) = 1$, therefore we get

$$\mu(e^\lambda - 1) = n_0\lambda.$$

From this relation we infer that

$$\frac{e^\lambda - 1}{\lambda} = \frac{n_0}{\mu} > 1. \tag{2.55}$$

Since the function $\frac{e^\lambda-1}{\lambda}$ is monoton increasing and goes to 1 as λ goes to 0, there exists a unique solution to 2.55. Unfortunately, there is no exact analytic expression for that solution. Now we calculate the constants appearing in Theorem 1. First we obtain the rescaling of the function:

$$\begin{aligned}\psi(u) = -\frac{1}{\lambda}\log\varphi(\lambda(1-u)) &= -\frac{1}{\lambda}\log e^{-n_0(\lambda(1-u))}e^{\mu(e^{\lambda(1-u)}-1)}\\ &= n_0(1-u) - \frac{\mu}{\lambda}(e^{\lambda(1-u)}-1).\end{aligned}$$

Moreover, using this result, we infer the next important constant,

$$\begin{aligned}I = \int_0^1 \psi(u)du &= \int_0^1 \left(n_0(1-u) - \frac{\mu}{\lambda}(e^{\lambda(1-u)}-1)\right)du\\ &= \frac{n_0}{2} + \frac{\mu}{\lambda^2}(1-e^\lambda) + \frac{\mu}{\lambda}.\end{aligned}$$

Finally, we get

$$\theta = 2\lambda\sqrt{\frac{n_0}{2} + \frac{\mu}{\lambda^2}(1-e^\lambda) + \frac{\mu}{\lambda}}.$$

Example 2.9. Let X be a random variable with distribution

$$\mathbb{P}(X=1) = 1 - \mathbb{P}(X=-1) = p \text{ with } p \in (0, \frac{1}{2}).$$

At first we obtain the moment generating function:

$$\varphi(t) = pe^t + (1-p)e^{-t}.$$

By the same arguments from the Cramér condition we infer the equation: $pe^t + (1-p)e^{-t} = 1$. In this case it is easy to find λ, that is obviously the solution of the quadratic equation $px^2 - x + (1-p) = 0$. Consequently $e^\lambda = \frac{1-p}{p}$. Taking into account, that $\lambda > 0$, we conclude that $p < \frac{1}{2}$. Then plugging the moment generating function, calculated above, into the definition of $\psi(u)$ we obtain

$$\psi(u) = -\frac{1}{\lambda}\log\left(pe^{\lambda(1-u)} + (1-p)e^{-\lambda(1-u)}\right),$$

and then one can easily get the next constant

$$I = \int_0^1 -\frac{1}{\lambda}\log\left(pe^{\lambda(1-u)} + (1-p)e^{-\lambda(1-u)}\right)du.$$

Using the substitution $v = 1-u$ we can apply the integration by part

$$\begin{aligned}-\frac{1}{\lambda}\int_0^1 \log(pe^{\lambda v} + (1-p)e^{-\lambda v})dv &= -\frac{1}{\lambda}v\log(pe^{\lambda v} + (1-p)e^{-\lambda v})\Big|_0^1 + \frac{1}{\lambda}\int_0^1 v\frac{p\lambda e^{\lambda v} - (1-p)\lambda e^{-\lambda v}}{pe^{\lambda v} + (1-p)e^{-\lambda v}}dv\\ &= \int_0^1 v\frac{pe^{\lambda v} - (1-p)e^{-\lambda v}}{pe^{\lambda v} + (1-p)e^{-\lambda v}}dv.\end{aligned}$$

Unfortunately in this case we do not know how to calculate the exact expression for I and consequently for θ neither.

Example 2.10. Now we look at the shifted geometric distribution

$$\mathbb{P}(X = n-1) = (1-p)p^n, \qquad n > 0.$$

One more time we beginn with the moment generating function

$$\varphi(t) = \frac{e^{-t}(1-p)}{(1-pe^t)}.$$

It is easy to see, that the Cramér condition $\varphi(\lambda) = 1$ leads us to the same equation for λ as in example 2.9:

$$\frac{1-p}{1-pe^\lambda} = e^\lambda.$$

Then by the same arguments as above we infer, that $p < \frac{1}{2}$ for $\lambda > 0$. Therefore we can derive the expression for the rescaling function

$$\psi(u) = -\frac{1}{\lambda}\log\frac{e^{-\lambda(u-1)}(1-p)}{(1-pe^{\lambda(u-1)})} = -\frac{1}{\lambda}\left(\lambda(u-1) + \log(1-p) - \log(1-pe^{\lambda(1-u)})\right).$$

In the next step we try to calculate the constant I

$$\begin{aligned} I &= \int_0^1 -(u-1)du - \int_0^1 \frac{1}{\lambda}\log(1-p)du + \int_0^1 \frac{1}{\lambda}\log(1-pe^{\lambda(1-u)})du \\ &= \frac{1}{2} - \frac{1}{\lambda}\log(1-p) + \int_0^1 \frac{1}{\lambda}\log(1-pe^{\lambda v})dv. \end{aligned}$$

Sorry to say, but this example left us also without success. As we have seen it is not easy to calculate the constants. In the most cases there is no calculative solution, only a numerical one.

Example 2.11. In this last example we will take the normal distribution with the negative mean $-a < 0$ and the variance $\theta^2 < \infty$. Using the same strategy, first we obtain the moment generating function

$$\varphi(t) = e^{-at+\sigma^2 t^2/2}.$$

Applying one more time the Cramér condition $\varphi(\lambda) = 1$, we get $-\lambda a + \frac{\sigma^2\lambda^2}{2} = 0$ and from this relation it is easy to see, that $\lambda = \frac{2a}{\sigma^2}$. Now combining this result with definition of the rescaling function, we obtain

$$\begin{aligned} \psi(u) = -\frac{1}{\lambda}\left(-a\left(\lambda(1-u)\right) + \frac{\sigma^2}{2}\lambda^2(1-u)^2\right) &= a(1-u) - \frac{\sigma^2}{2}\lambda(1-u)^2 \\ &= au(1-u). \end{aligned}$$

Using this result we get this time a very nice integral for the parameter I, namely

$$I = \int_0^1 au(1-u)du = \frac{a}{6}.$$

Finally we can obtain also the parameter θ

$$\theta = \sqrt{\frac{8}{3}\frac{a^3/2}{\sigma^2}}.$$

Unfortunately for the continious case we can not use the Theorem 2.1, but at least the parameter θ can be used to compute an upper bound for $\mathbf{P}(A_\tau > x)$ via the exponential Chebyshev inequality, more precisely

$$\mathbf{P}(A_\tau > x) \le Cx^{1/4} \exp\left\{-\sqrt{\frac{8}{3}\frac{a^3/2}{\sigma^2}}\sqrt{x}\right\}. \tag{2.56}$$

3 Local tail asymptotics for the joint distribution of length and maximum of a random walk excursion

This section is the subject of the article [29] written in collaboration with Vitali Wachtel.

This section is devoted to the study of the maximum of the excursion of a random walk with negative drift and light-tailed increments. More precisely, we determine the local asymptotics of the joint distribution of the length, maximum and the time at which this maximum is achieved. This result allows one to obtain a local central limit theorem for the length of the excursion conditioned on large values of the maximum.

3.1 Introduction and statement of main results

Let $\{S_n; n \geq 1\}$ be a random walk with i.i.d. increments $\{X_k; k \geq 1\}$. We are interested in the asymptotic properties of some functionals of the excursion

$$\{S_1, S_2, \dots, S_\tau\},$$

where

$$\tau := \inf\{n \geq 1 : S_n \leq 0\}.$$

Clearly, τ is almost surely finite for all random walks satisfying $\liminf_{n\to\infty} S_n = -\infty$. This holds, in particular, for all random walks with non-positive drift.
Denote

$$M_n := \max_{k \leq n} S_k, \quad n \geq 1.$$

The main purpose of this note is to determine the asymptotic behaviour of local probabilities of the vector $(M_\tau, \theta_\tau, \tau)$, where

$$\theta_\tau := \min\{n \geq 0 : S_n = M_\tau\}.$$

We shall always assume that S_n is integer-valued, has negative drift, and satisfies the Cramér condition: there exists $\lambda > 0$ such that

$$\varphi(\lambda) := \mathbf{E} e^{\lambda X_1} = 1. \tag{3.1}$$

Obviously, (3.1) implies that

$$\mathbf{E}[X_1] =: -a < 0.$$

It is well-known, see Iglehart [?], that if, additionally, $\mathbf{E}[X_1 e^{\lambda X_1}] < \infty$ then there exists $c_0 \in (0,1)$ such that, as $x \to \infty$,

$$\mathbf{P}(M_\tau = x) \sim c_0 e^{-\lambda x}, \tag{3.2}$$

provided that the distribution of X_1 is aperiodic.
Moreover, according to Theorem II in Doney [13], one has, for all aperiodic walks satisfying (3.1),

$$\mathbf{P}(\tau = n) \sim c_1 n^{-3/2} \left(\mathbf{E} e^{\mu X_1}\right)^n, \tag{3.3}$$

where $\mu > 0$ is uniquely determined by $\mathbf{E}[X_1 e^{\mu X_1}] = 0$.
These marginal asymptotics do not allow one to guess the right asymptotic behaviour of the joint distribution. The reason is a very strong dependence between large values of τ and M_τ. This can be illustrated by the optimal strategy for the occurence of the event $\{M_\tau = x\}$. First, the random walk goes up linearly with the rate

$$\widehat{a} := \mathbf{E}[X_1 e^{\lambda X_1}]/\mathbf{E}[e^{\lambda X_1}].$$

After reaching the level x, the random walk goes down with the standard rate a. As a result, the stopping time τ is of the order $x/\widehat{a} + x/a$ on $\{M_\tau > x\}$.
According to Theorem 5.2 in Asmussen [1], conditioned on the large value of the maximum M_τ, θ_τ satisfies a central limit theorem. More precisely,

$$\mathbf{P}\left(\frac{\theta_\tau - x/\widehat{a}}{\widehat{c}\sqrt{x}} < u \middle| M_\tau > x\right) \to \Phi(u), \quad x \to \infty, \tag{3.4}$$

where

$$\widehat{c} := \sqrt{\frac{\widehat{\sigma}^2}{\widehat{a}^3}} \quad \text{and} \quad \widehat{\sigma}^2 := \mathbf{E}[X_1^2 e^{\lambda X_1}] - \widehat{a}^2.$$

Clearly, this result improves the description of the first part of the optimal strategy leading to $\{M_\tau > x\}$.
Our main result describes the asymptotic behaviour of the mass function of the vector $(M_\tau, \theta_\tau, \tau)$.

Theorem 3.1. *Assume that (3.1) holds and, furthermore,*

$$\widehat{\sigma}^2 = \mathbf{E}[X_1^2 e^{\lambda X_1}] - \widehat{a}^2 \in (0, \infty).$$

Assume also that the distribution of X_1 is aperiodic. Then there exists $Q > 0$ such that, as $x \to \infty$,

$$e^{\lambda x}\mathbb{P}(M_\tau = x, \theta_\tau = k, \tau = n+1) = \frac{Q}{2\pi\sqrt{k(n-k)}} \exp\left\{-\frac{(x-k\widehat{a})^2}{2\widehat{\sigma}^2 k} - \frac{(x-a(n-k))^2}{2\sigma^2(n-k)}\right\} + o\left(\frac{1}{\sqrt{k(n-k)}}\right)$$

uniformly in k, n such that $k < n$. The exact form of the constant Q is given in (3.51).

The proof strategy in Theorem 3.1 enables one to prove local limit theorems for τ and θ_τ conditioned on $M_\tau = x$.

Corollary 3.2. *Under the conditions of Theorem 3.1, as $x \to \infty$,*

$$\sqrt{x}\mathbf{P}(\theta_\tau = k | M_\tau = x) = \frac{\widehat{a}^{3/2}}{\sqrt{2\pi\widehat{\sigma}^2}} \exp\left\{-\frac{\widehat{a}^3(k - x/\widehat{a})^2}{2x\widehat{\sigma}^2}\right\} + o(1) \tag{3.5}$$

and

$$\sqrt{x}\mathbf{P}(\tau - \theta_\tau = k | M_\tau = x) = \frac{a^{3/2}}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{a^3(k - x/a)^2}{2x\sigma^2}\right\} + o(1) \tag{3.6}$$

uniformly in k. Furthermore, conditionally on $M_\tau = x$, the random variables θ_τ and $\tau-\theta_\tau$ are independent. Consequently,

$$\sqrt{x}\mathbf{P}(\tau = n|M_\tau = x) = \frac{1}{\sqrt{2\pi\Sigma^2}}\exp\left\{-\frac{(n-Ax)^2}{2x\Sigma^2}\right\} + o(1), \tag{3.7}$$

where

$$A := \frac{1}{a} + \frac{1}{\widehat{a}} \quad \text{and} \quad \Sigma^2 := \frac{\sigma^2}{a^3} + \frac{\widehat{\sigma}^2}{\widehat{a}^3}.$$

Since the tail of M_τ is exponentially decreasing, one infers from (3.4) that

$$\mathbf{P}\left(\frac{\theta_\tau - x/\widehat{a}}{\widehat{c}\sqrt{x}} < u \Big| M_\tau = x\right) \to \Phi(u), \quad x \to \infty. \tag{3.8}$$

Thus, (3.5) is a local version of this central limit theorem for θ_τ.
Our approach to the excursions of random walks satisfying the Cramér condition (3.1) is based on the standard change of measure:

$$\widehat{\mathbf{P}}(X_1 \in dx) = e^{\lambda x}\mathbf{P}(X_1 \in dx).$$

Under this new measure the drift of S_n becomes positive: $\widehat{\mathbf{E}}X_1 = \widehat{a} > 0$. Furthermore, the variance of X_k under the new measure becomes equal to $\widehat{\sigma}^2$. For that reason we study in the next section local probability asymptotics for conditioned random walks with positive drift. These results are later used in the proof of Theorem 3.1, which is given in Section 3.

3.2 Local limit theorems for functionals of a random walk with positive drift

In this section we shall always assume that $\mathbf{E}X_1 = a > 0$ and $\mathbf{E}(X_1 - a)^2 = \sigma^2 \in (0, \infty)$.

3.2.1 Local limit theorem for a walk conditioned to stay positive

Define the stopping times

$$\tau_z = \inf\{n \ge 1 : z + S_n \le 0\}, \quad z \ge 0.$$

In the case of random walks with positive drift one has $\mathbf{P}(\tau_z = \infty) > 0$ and, consequently, the asymptotic behaviour of S_n conditioned to stay positive should be the same as for the unconditioned walk. In the case $z = 0$ Iglehart [21] has shown that for conditioned random walk the standard form of the functional central limit theorem is still valid, see Proposition 2.1 in [21]. Our first result shows that CLT for marginals holds for all starting points z.

Proposition 3.3. *Assume that $a > 0$ and that $\sigma^2 \in (0, \infty)$. Then, for every $z \ge 0$,*

$$\mathbb{P}\left(\frac{S_n - na}{\sigma\sqrt{n}} \le x \Big| \tau_z > n\right) \to \Phi(x), \quad x \in \mathbb{R}, \tag{3.9}$$

where $\Phi(x)$ is the the standard normal distribution function.

Proof.
As in the proof of Proposition 2.1 from [21], we shall use the decomposition

$$\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x,\tau_z>n\right)$$
$$=\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x\right)-\sum_{k=1}^{n}\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x,\tau_z=k\right). \tag{3.10}$$

By the central limit theorem,

$$\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x\right)\to\Phi(x). \tag{3.11}$$

By the Markov property, for every fixed k,

$$\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x,\tau_z=k\right)=\int_{-\infty}^{-z}\mathbb{P}(S_k\in dy,\tau_z=k)\mathbb{P}\left(\frac{y+S_{n-k}-na}{\sigma\sqrt{n}}\le x\right).$$

Then, using (3.11) and the dominated convergence, we obtain

$$\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x,\tau_z=k\right)\to\Phi(x)\mathbb{P}(\tau_z=k).$$

Consequently, for every fixed $N>1$,

$$\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x\right)-\sum_{k=1}^{N}\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x,\tau_z=k\right)$$
$$\to\Phi(x)\left(1-\sum_{k=1}^{N}\mathbb{P}(\tau_z=k)\right)=\Phi(x)\mathbb{P}(\tau_z>N). \tag{3.12}$$

For the tail of the sum one has, uniformly in $n>N$,

$$0<\sum_{k=N}^{n}\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x,\tau_z=k\right)\le\sum_{k=N}^{n}\mathbb{P}(\tau_z=k)$$
$$=\mathbb{P}(N\le\tau_z\le n)\le\mathbb{P}(N\le\tau_z<\infty). \tag{3.13}$$

Combining (3.12) and (3.13) and letting $N\to\infty$, we obtain

$$\mathbb{P}\left(\frac{S_n-na}{\sigma\sqrt{n}}\le x,\tau_z>n\right)\to\Phi(z)\mathbb{P}(\tau_z=\infty).$$

The positivity of the drift implies that $\mathbb{P}(\tau_z=\infty)$ is positive. Therefore, the previous convergence is equivalent to the statement of the proposition. □

We now turn to the corresponding local limit theorem.

Theorem 3.4. *Assume that $a>0$ and that $\sigma^2\in(0,\infty)$. Assume also, that the distribution of X_1 is aperiodic. Then, uniformly in $x>-z$,*

$$\mathbb{P}(S_n=x,\tau_z>n)=\frac{\mathbf{P}(\tau_z=\infty)}{\sqrt{2\pi\sigma^2n}}e^{-(x-na)^2/2\sigma^2n}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.14}$$

Proof.
Set $m = \left[\frac{n}{2}\right]$. By the Markov property,

$$\begin{aligned}
&\mathbb{P}(S_n = x, \tau_z > n) = \\
&\quad = \sum_{y=1-z}^{\infty} \mathbb{P}\left(S_m = y, \tau_z > m\right) \mathbb{P}\left(S_{n-m} = x - y, \min_{k \le n-m} S_k \ge -y - z\right).
\end{aligned} \tag{3.15}$$

We split the sum in (3.15) into two parts. First,

$$\begin{aligned}
&\sum_{y=1-z}^{am/2} \mathbb{P}(S_m = y, \tau_z > m)\mathbb{P}(S_{n-m} = x - y, \min_{k \le x-y} S_k \ge -y - z) \\
&\qquad\qquad \le \sup_u \mathbb{P}(S_{n-m} = u)\mathbb{P}\left(S_m \le \frac{am}{2}, \tau_z > m\right).
\end{aligned} \tag{3.16}$$

Using the Chebyshev inequality we infer that

$$\mathbb{P}\left(S_m \le \frac{am}{2}, \tau_z > m\right) \le \mathbb{P}(S_m \le \frac{am}{2}) \to 0, \quad \text{as } m \to \infty. \tag{3.17}$$

Furthermore, by the local limit theorem for unconditioned random walks,

$$\sup_z \mathbb{P}(S_{n-m} = z) \le \frac{c}{\sqrt{n-m}}. \tag{3.18}$$

Plugging this estimate into (3.16) and using (3.17), we get

$$\sum_{y=1-z}^{am/2} \mathbb{P}(S_m = y, \tau_z > m)\mathbb{P}(S_{n-m} = x - y, \min_{k \le n-m} S_k \ge -y - z) = o\left(\frac{1}{\sqrt{n}}\right). \tag{3.19}$$

Second, for $y > am/2$ we shall use the representation

$$\begin{aligned}
&\mathbb{P}\left(S_{n-m} = x - y, \min_{k \le n-m} S_k \ge -y - z\right) \\
&\qquad = \mathbb{P}\left(S_{n-m} = x - y\right) - \mathbb{P}\left(S_{n-m} = x - y, \min_{k \le x-y} S_k < -y - z\right).
\end{aligned}$$

Therefore,

$$\mathbb{P}\left(S_{n-m} = x - y, \min_{k \le n-m} S_k \ge -y - z\right) \le \mathbb{P}(S_{n-m} = x - y),$$

and

$$\begin{aligned}
&\mathbb{P}\left(S_{n-m} = x - y, \min_{k \le n-m} S_k \ge -y - z\right) \\
&\qquad\qquad \ge \mathbb{P}\left(S_{n-m} = x - y\right) - \mathbb{P}\left(\min_{k \le n-m} S_k < -\frac{am}{2}\right).
\end{aligned}$$

Applying the classical Kolmogorov inequality, we get

$$\begin{aligned}\mathbb{P}\left(\min_{k\le n-m} S_k < -\frac{am}{2}\right) &\le \mathbb{P}\left(\min_{k\le n-m}(S_k - \mathbf{E}[S_k]) < -\frac{am}{2}\right)\\ &\le \frac{\mathbf{Var}[S_{n-m}]}{a^2m^2/4} \le \frac{c}{n}.\end{aligned} \tag{3.20}$$

Consequently,

$$\begin{aligned}&\sum_{y\ge\frac{am}{2}} \mathbb{P}(S_m = y, \tau_z > m)\mathbb{P}\left(S_{n-m} = x-y, \min_{k\le n-m} S_k > -y-z\right) =\\ &= \sum_{y\ge\frac{am}{2}} \mathbb{P}(S_m = y, \tau_z > m)\mathbb{P}(S_{n-m} = x-y) + O\left(\frac{1}{n}\right).\end{aligned} \tag{3.21}$$

Combining (3.19) and (3.21) we get

$$\mathbb{P}(S_n = x, \tau_z > n) = \sum_{y=1}^{\infty} \mathbb{P}(S_m = y, \tau_z > m)\mathbb{P}(S_{n-m} = x-y) + o\left(\frac{1}{\sqrt{n}}\right).$$

By the local limit theorem for $\{S_n\}$,

$$\mathbb{P}(S_{n-m} = x-y) = \frac{1}{\sqrt{2\pi(n-m)\sigma^2}} e^{-(x-y-a(n-m))^2/2\sigma^2(n-m)} + o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly in $x-y$.
Therefore,

$$\begin{aligned}&\mathbb{P}(S_n = x, \tau_z > n)\\ &\quad= \frac{\mathbb{P}(\tau_z > m)}{\sqrt{2\pi(n-m)\sigma^2}}\mathbf{E}\left[e^{-(x-S_m-an+am)^2/2\sigma^2(n-m)}|\tau_z > m\right] + o\left(\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{3.22}$$

Using now Proposition 3.3, we get

$$\begin{aligned}&\mathbf{E}\left[e^{-(x-an-(S_m-am))^2/2(n-m)\sigma^2}|\tau_z > m\right]\\ &\qquad= \int_{-\infty}^{\infty} e^{-\left(\frac{x-an}{\sqrt{(n-m)\sigma^2}}-u\right)^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} + o(1)\\ &\qquad= \frac{1}{\sqrt{2}} e^{-(x-an)^2/2n\sigma^2} + o(1).\end{aligned} \tag{3.23}$$

Plugging (3.23) into (3.22), we obtain

$$\mathbb{P}(S_n = x, \tau_z > n) = \frac{\mathbb{P}(\tau_z > m)}{\sqrt{2\pi\sigma^2 n}} e^{-(x-an)^2/2n\sigma^2} + o\left(\frac{1}{\sqrt{n}}\right).$$

Recalling that $\mathbb{P}(\tau_z > m) \to \mathbb{P}(\tau_z = \infty)$, we finally get the relation (3.14). $\square$

3.3 Local asymptotics for (M_n, S_n)

Define

$$\tau_+ := \min\{n \geq 1 : S_n > 0\}.$$

Theorem 3.5. *Under the assumptions of Theorem 3.4, uniformly in $x \geq 0$ and $r \geq 0$,*

$$\mathbb{P}(M_n = x, S_n = x - r) = \frac{\mathbf{P}(\tau_+ = \infty)V(r)}{\sqrt{2\pi\sigma^2 n}} e^{-(x-na)^2/2\sigma^2 n} + o\left(\frac{1}{\sqrt{n}}\right),$$

where

$$V(r) := \sum_{j=0}^{\infty} \mathbb{P}(S_j = -r, \tau_+ > j).$$

Proof.
Let θ_n be the first time the random walk achieves its maximum. That is,

$$\theta_n := \min\{k \geq 1 : S_k = M_n\}.$$

By the Markov property,

$$\begin{aligned}
&\mathbb{P}(M_n = x, S_n = x - r) \\
&\quad = \sum_{k=0}^{n} \mathbb{P}(M_n = x, S_n = x - r, \theta_n = k) \\
&\quad = \sum_{k=0}^{n} \mathbb{P}(S_k = x, S_j < S_k \text{ for all } j < k)\mathbb{P}(S_{n-k} = -r, S_j \leq 0 \text{ for all } j \leq n - k) \\
&\quad = \sum_{k=0}^{n} \mathbb{P}(S_k = x, S_k - S_j > 0 \text{ for all } j < k)\mathbb{P}(S_{n-k} = -r, \tau_+ > n - k). \qquad (3.24)
\end{aligned}$$

It follows from the duality lemma for random walks that

$$\mathbb{P}(S_k = x, S_k - S_j > 0 \text{ for all } j < k) = \mathbb{P}(S_k = x, \tau > k). \qquad (3.25)$$

Combining (3.24) and (3.25), we obtain

$$\mathbb{P}(M_n = x, S_n = x - r) = \sum_{k=0}^{n} \mathbb{P}(S_k = x, \tau > k)\mathbb{P}(S_{n-k} = -r, \tau_+ > n - k). \qquad (3.26)$$

Fix some $\varepsilon \in (0, 1/2)$. The assumption $a > 0$ implies that $\mathbf{E}\tau_+$ is finite. Therefore, uniformly in $k \leq (1-\varepsilon)n$ and $r \geq 0$,

$$\mathbb{P}(S_{n-k} = -r, \tau_+ > n - k) \leq \mathbb{P}(\tau_+ > \varepsilon n) = o\left(\frac{1}{n}\right).$$

Consequently, using the transience of $\{S_n\}$,

$$\begin{aligned}
&\sum_{k=0}^{(1-\varepsilon)n} \mathbb{P}(S_k = x, \tau > k)\mathbb{P}(S_{n-k} = -r, \tau_+ > n - k) \\
&\qquad\qquad \leq \mathbb{P}(\tau_+ > \varepsilon n) \sum_{k=0}^{\infty} \mathbb{P}(S_k = x) = o\left(\frac{1}{n}\right). \qquad (3.27)
\end{aligned}$$

If $k \geq (1-\varepsilon)n$ then, by (3.18),

$$\mathbb{P}(S_k = x, \tau > k) \leq \mathbb{P}(S_k = x) \leq \frac{c}{\sqrt{(1-\varepsilon)n}} \leq \frac{2c}{\sqrt{n}}. \tag{3.28}$$

This bound implies that, for every fixed N,

$$\begin{aligned}\sum_{(1-\varepsilon)n}^{n-N} &\mathbb{P}(S_k = x, \tau > k)\mathbb{P}(S_{n-k} = -r, \tau_+ > n-k) \\ &\leq \frac{c}{\sqrt{n}} \sum_{j=N}^{\varepsilon n} \mathbb{P}(S_j = -r, \tau_+ > j) \leq \frac{c}{\sqrt{n}} \sum_{j=N}^{\infty} \mathbb{P}(\tau_+ > j).\end{aligned} \tag{3.29}$$

We also note that the finiteness of $\mathbf{E}\tau_+$ yields

$$\lim_{N\to\infty} \sum_{j=N}^{\infty} \mathbb{P}(\tau_+ > j) = 0. \tag{3.30}$$

Using Theorem 3.4 with $z = 0$, we obtain

$$\begin{aligned}\sum_{k=n-N}^{n} &\mathbb{P}(S_k = x, \tau > k)\mathbb{P}(S_{n-k} = -r, \tau_+ > n-k) \\ &= \frac{\mathbf{P}(\tau = \infty)}{\sqrt{2\pi\sigma^2 n}} e^{-(x-na)^2/2\sigma^2 n} \sum_{j=0}^{N} \mathbb{P}(S_j = -r, \tau_+ > j) + o\left(\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{3.31}$$

Plugging (3.27), (3.29) and (3.31) into (3.26), we obtain

$$\begin{aligned}&\left|\mathbb{P}(M_n = x, S_n = x-r) - \frac{\mathbb{P}(\tau = \infty)}{\sqrt{2\pi\sigma^2 n}} e^{(x-na)^2/2\sigma n} \sum_{j=0}^{N} \mathbb{P}(S_j = -r, \tau_+ > j)\right| \\ &\qquad \leq \sum_{j=N}^{\infty} \mathbb{P}(\tau_+ > j) + o\left(\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{3.32}$$

Letting now $N \to \infty$ and taking into account (3.30), we arrive at (3.32). □

Corollary 3.6. *Under the conditions of Theorem 3.4,*

$$\mathbf{P}(M_n = x) = \frac{1}{\sqrt{2\pi\sigma^2 n}} e^{-(x-an)^2/2n} + o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly in $x \geq 0$.

Proof.
Similar to the proof of Theorem 3.5,

$$\begin{aligned}\mathbb{P}(M_n = x) &= \sum_{k=0}^{n} \mathbb{P}(M_n = x, \theta_n = k) \\ &= \sum_{k=0}^{n} \mathbb{P}(S_k = x, S_k - S_j > 0 \text{ for all } j < k)\mathbb{P}(\tau_+ > n-k) \\ &= \sum_{k=0}^{n} \mathbb{P}(S_k = x, \tau > k)\mathbb{P}(\tau_+ > n-k).\end{aligned} \tag{3.33}$$

The bound

$$\mathbb{P}(\tau_+ > \varepsilon n) = o\left(\frac{1}{n}\right),$$

implies that

$$\sum_{k=0}^{(1-\varepsilon)n} \mathbb{P}(S_k = x, \tau > k)\mathbb{P}(\tau_+ > n-k)$$
$$\leq \mathbb{P}(\tau_+ > \varepsilon n) \sum_{k=0}^{\infty} \mathbb{P}(S_k = x) = o\left(\frac{1}{n}\right). \tag{3.34}$$

In the last step we have also used the fact that $\{S_k\}$ is transient. Using the same arguments as in the derivation of (3.28),

$$\sum_{(1-\varepsilon)}^{n-N} \mathbb{P}(S_k = x, \tau > k)\mathbb{P}(\tau_+ > n-k) \leq \frac{c}{\sqrt{n}} \sum_{j=N}^{\infty} \mathbb{P}(\tau_+ > j) = \frac{\varepsilon_N}{\sqrt{n}}. \tag{3.35}$$

Finally, using (3.14), we obtain

$$\sum_{k=n-N}^{n} \mathbb{P}(S_k = x, \tau > k)\mathbb{P}(\tau_+ > n-k)$$
$$= \frac{\mathbf{P}(\tau = \infty)}{\sqrt{2\pi\sigma^2 n}} e^{-(x-na)^2/2\sigma^2 n} \sum_{j=0}^{N} \mathbb{P}(\tau_+ > j). \tag{3.36}$$

Combining (3.33),(3.34), (3.35) and (3.36) and letting $N \to \infty$, we conclude that

$$\mathbb{P}(M_n = x) = \frac{\mathbf{P}(\tau = \infty)\mathbf{E}\tau_+}{\sqrt{2\pi\sigma^2 n}} e^{-(x-na)^2/2\sigma^2 n} + \left(\frac{1}{\sqrt{n}}\right).$$

Noting that the duality of stopping times τ and τ_+ implies that

$$(1 - \mathbf{E}z^{\tau_+})(1 - \mathbf{E}z^{\tau}) = 1 - z.$$

Dividing both parts by $1 - z$ and letting $z \to 1$, we get

$$\mathbf{P}(\tau = \infty)\mathbf{E}\tau_+ = 1. \tag{3.37}$$

This equality completes the proof. □

Corollary 3.7. *If $x - na = O(\sqrt{n})$, then for every fixed $r \geq 0$*

$$\lim_{n\to\infty} \mathbb{P}(S_n = x - r | M_n = x) = \mathbf{P}(\tau_z = \infty)V(r) = \frac{V(r)}{\mathbf{E}\tau_+}. \tag{3.38}$$

Proof.
Combining Theorem 3.5 and Corollary 3.6 and using (3.37), we obtain the desired relation. □

Consider the sequence

$$R_n := M_n - S_n, \quad n \ge 0.$$

It is well-known and easy to see that this sequence can be defined by

$$R_{n+1} = (R_n - X_{n+1})^+.$$

Furthermore, for every n the distribution of R_n is equal to that of $\max_{k\le n}(-S_k)$. The assumption $a > 0$ implies now that, as $n \to \infty$,

$$\mathbf{P}(M_n - S_n = r) = \mathbf{P}(R_n = r) \to \mathbf{P}\left(\max_{k\ge 1}(-S_k) = r\right), \quad r \ge 0.$$

By the ladder heights representation for $\max_{k\ge1}(-S_k)$ and by the duality lemma,

$$\begin{aligned}\mathbf{P}\left(\max_{k\ge 1}(-S_k) = r\right) &= \sum_{n=0}^{\infty} \mathbb{P}(S_n = -r,\ n \text{ is a descending ladder epoch})\mathbb{P}(\tau = \infty)\\ &= \sum_{j=0}^{\infty} \mathbb{P}(S_j = -r, \tau_+ > j)\mathbb{P}(\tau = \infty) = \frac{V(r)}{\mathbf{E}\tau_+}.\end{aligned}$$

As a result, as $n \to \infty$,

$$\mathbb{P}(M_n - S_n = r) \to \frac{V(r)}{\mathbf{E}\tau_+}$$

for every $r \ge 0$. Obviously, this classical relation is a consequence of our Corollary 3.7.

Theorem 3.8. *Under the conditions of Theorem 3.4, for fixed non-negative numbers y, z,*

$$\begin{aligned}&\mathbb{P}(S_n = x, M_{n-1} < x + y, \tau_z > n)\\ &\qquad = \frac{\mathbb{P}(\tau_y = \infty)\mathbb{P}(\tau_z = \infty)}{\sqrt{2\pi\sigma^2 n}} e^{-(x-na)^2/2\sigma^2 n} + o\left(\frac{1}{\sqrt{n+x}}\right)\end{aligned}$$

uniformly in x.

Proof.
Fix $\varepsilon > 0$. If $|x - na| > n\varepsilon$ then by the Chebyshev inequality

$$\mathbb{P}(S_n = x, M_{n-1} < x + y, \tau_z > n) \le \mathbb{P}\left(|S_n - an| \ge |x - an|\right) = o\left(\frac{1}{n+x}\right). \tag{3.39}$$

Thus, it remains to consider the case $|x - na| \le \varepsilon n$. Set again $m = [n/2]$. By the Markov property,

$$\begin{aligned}&\mathbb{P}(S_n = x, M_{n-1} < x + y, \tau_z > n)\\ &= \sum_{u=1-z}^{x+y-1} \mathbb{P}(S_m = u, M_m < x + y, \tau_z > m)\\ &\qquad\qquad \times \mathbb{P}(S_{n-m} = x - u, M_{n-m} < x + y - u, \tau_{z+u} > n - m)\\ &= \Sigma_1 + \Sigma_2,\end{aligned} \tag{3.40}$$

where Σ_1 is the sum over $u \in (-z, [an/4]]$, and Σ_2 over $u \in ([an/4]+1, x+y-1)$. Using the Chebyshev inequality once again, we obtain

$$\Sigma_1 \leq \mathbb{P}\left(S_m \leq \left[\frac{an}{4}\right]\right) = o\left(\frac{1}{n}\right).$$

Therefore,

$$\mathbb{P}(S_n = x, M_{n-1} < x+y, \tau_z > n) = \Sigma_2 + o\left(\frac{1}{n}\right).$$

By the Kolmogorov inequality, for $x \geq (a-\varepsilon)n$ and $y \geq 0$,

$$\mathbb{P}(M_m \geq x+y) = o\left(\frac{1}{n}\right).$$

Consequently,

$$\begin{aligned}
&\mathbb{P}(S_m = u, M_m < x+y, \tau_z > m) \\
&\qquad = \mathbb{P}(S_m = u, \tau_z > m) - \mathbb{P}(S_m = u, M_m \geq x+y, \tau > m) \\
&\qquad = \mathbb{P}(S_m = u, \tau_z > m) + o\left(\frac{1}{n}\right).
\end{aligned} \tag{3.41}$$

Using once again the Kolmogorov inequality, we get

$$\mathbf{P}(\tau_{z+u} < n-m) = \mathbb{P}\left(\min_{k \leq n-m} S_k < -z-u\right) = o\left(\frac{1}{n}\right)$$

uniformly in $u \geq an/4$.
Therefore,

$$\begin{aligned}
&\mathbb{P}(S_{n-m} = x-u, M_{n-m} < x+y-u, \tau_{z+u} > n-m) = \\
&\qquad \mathbb{P}(S_{n-m} = x-u, M_{n-m} < x+y-u) + o\left(\frac{1}{n}\right).
\end{aligned} \tag{3.42}$$

Furthermore, by the duality lemma,

$$\mathbb{P}(S_{n-m} = x-u, M_{n-m} < x+y-u) = \mathbb{P}(S_{n-m} = x-u, \tau_y > n-m).$$

Plugging these equalities into Σ_2, we conclude that

$$\Sigma_2 = \sum_{u=[an/4]+1}^{x+y-1} \mathbb{P}(S_m = u, \tau_z > m)\mathbb{P}(S_{n-m} = x-u, \tau_y > n-m) + o\left(\frac{1}{n}\right).$$

Applying now Theorem 3.4, we finally get

$$\begin{aligned}
\Sigma_2 &= \frac{\mathbb{P}(\tau_z = \infty)\mathbb{P}(\tau_y = \infty)}{2\pi\sigma^2\frac{n}{2}} \sum_{u=[an/4]}^{x+y-1} e^{-(u-am)^2/2\sigma^2 m} e^{-(x-u-a(n-m))^2/2\sigma^2(n-m)} \\
&\qquad + o\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{\mathbb{P}(\tau_z = \infty)\mathbb{P}(\tau_y = \infty)}{\sqrt{2\pi\sigma^2 n}} e^{-(x-an)^2/2\sigma^2 n} + o\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

This completes the proof. $\square$

3.4 Proofs of main results.

3.4.1 Proof of Theorem 3.1

Clearly,

$$\begin{aligned}
&\mathbb{P}(M_\tau = x, \theta_\tau = k, \tau = n+1) \\
&\qquad = \mathbb{P}(M_n = x, \theta_n = k, \tau = n+1) \\
&\qquad = \sum_{y=1}^{x} \mathbb{P}(M_n = x, \theta_n = k, S_n = y, \tau > n)\mathbb{P}(X_{n+1} \le -y). \qquad (3.43)
\end{aligned}$$

Furthermore, for every $y \in \{1, 2 \cdots, x\}$ we have

$$\begin{aligned}
&\mathbb{P}(M_n = x, \theta_n = k, S_n = y, \tau > n) \\
&\qquad = \mathbb{P}(S_k = x, \theta_k = k, \tau > k)\mathbb{P}(S_{n-k} = y - x, M_{n-k} \le 0, \min_{j \le n-k} S_j > -x). \qquad (3.44)
\end{aligned}$$

Consider a new measure $\widehat{\mathbb{P}}$ given by

$$\widehat{\mathbb{P}}(X_k \in du) = \frac{e^{\lambda u}}{\varphi(\lambda)}\mathbb{P}(X_k \in du), \quad k \ge 1.$$

Then one has

$$\begin{aligned}
\mathbb{P}(S_k = x, \theta_k = k, \tau > k) &= e^{-\lambda x}\widehat{\mathbb{P}}(S_k = x, \theta_k = k, \tau > k) \\
&= e^{-\lambda x}\widehat{\mathbb{P}}(S_k = x, M_{k-1} < x, \tau > k). \qquad (3.45)
\end{aligned}$$

Applying Theorem 3.8 with $z = y = 0$, we obtain, uniformly in x,

$$e^{\lambda x}\mathbb{P}(S_k = x, \theta_k = k, \tau > k) = \frac{\widehat{\mathbb{P}}^2(\tau = \infty)}{\sqrt{2\pi\widehat{\sigma}^2 k}} e^{-(x - k\widehat{a})^2/2\widehat{\sigma}^2 k} + o\left(\frac{1}{\sqrt{x+k}}\right). \qquad (3.46)$$

For every $k \ge 0$ set $\overline{S}_k = -S_k$. Then

$$\begin{aligned}
&\mathbb{P}(S_{n-k} = y - x, M_{n-k} \le 0, \min_{j \le n-k} S_j > -x) \\
&\qquad = \mathbb{P}(\overline{S}_{n-k} = x - y, \min_{j \le n-k} \overline{S}_j \ge 0, \max_{j \le n-k} \overline{S}_j < x) \\
&\qquad = \mathbb{P}(\overline{S}_{n-k} = x - y, \overline{\tau}_1 > n - k, \overline{M}_{n-k} < x). \qquad (3.47)
\end{aligned}$$

Applying Theorem 3.8 to the random walk $\{\overline{S}_n\}$, we get, for every fixed y,

$$\begin{aligned}
&\mathbb{P}(\bar{S}_{n-k} = x - y, \bar{\tau}_1 > n - k, \bar{M}_{n-k} < x) \\
&\qquad = \frac{\mathbb{P}(\overline{\tau}_1 = \infty)\mathbb{P}(\overline{\tau}_y = \infty)}{\sqrt{2\pi\sigma^2(n-k)}} e^{-(x - a(n-k))^2/2\sigma^2(n-k)} + o\left(\frac{1}{\sqrt{x + (n-k)}}\right). \qquad (3.48)
\end{aligned}$$

Furthermore, by (3.18),

$$\widehat{\mathbb{P}}(S_k = x, \theta_n = k, \tau > k) \le \widehat{\mathbb{P}}(S_k = x) \le \frac{c}{\sqrt{k}}$$

and

$$\mathbb{P}(\overline{S}_{n-k} = x - y, \overline{\tau}_1 > n - k, \overline{M}_{n-k} < x) \le \mathbb{P}(\overline{S}_{n-k} = x - y) \le \frac{c}{\sqrt{n-k}}.$$

Combining these estimates with (3.44) and (3.47), we obtain

$$\begin{aligned} e^{\lambda x} \sum_{y=N+1}^{x} \mathbb{P}(M_n = x, \theta_n = k, \tau > n)\mathbb{P}(X_{n+1} \le -y) \\ \le \frac{c}{\sqrt{k(n-k)}} \sum_{y=N+1}^{\infty} \mathbb{P}(X_{n+1} \le -y). \end{aligned} \tag{3.49}$$

Combining (3.46) and (3.48), we conclude that

$$\begin{aligned} e^{\lambda x} & \sum_{y=1}^{N} \mathbb{P}(M_n = x, S_n = y, \theta_n = k, \tau > n)\mathbb{P}(X_{n+1} \le -y) \\ &= \frac{\widehat{\mathbb{P}}^2(\tau = \infty)\mathbb{P}(\overline{\tau}_1 = \infty)}{2\pi\sqrt{\sigma^2\widehat{\sigma}^2 k(n-k)}} \exp\left\{-\frac{(x-k\widehat{a})^2}{2\widehat{\sigma}^2 k} - \frac{(x-a(n-k))^2}{2\sigma^2(n-k)}\right\} \Sigma_N \\ &\quad + o\left(\frac{1}{\sqrt{k(n-k)}}\right), \end{aligned} \tag{3.50}$$

$$\Sigma_N := \sum_{y=1}^{N} \mathbb{P}(\overline{\tau}_y = \infty)\mathbb{P}(X_1 \le -y).$$

Clearly,

$$\Sigma_N \to \sum_{y=1}^{\infty} \mathbb{P}(\overline{\tau}_y = \infty)\mathbb{P}(X_1 \le -y) \quad \text{as } N \to \infty.$$

Plugging now (3.50) and (3.49) into (3.43) and letting $N \to \infty$, we conclude that

$$\begin{aligned} & e^{\lambda x}\mathbb{P}(M_\tau = x, \theta_\tau = k, \tau = n+1) = \\ & \frac{Q}{2\pi\sqrt{k(n-k)}} \exp\left\{-\frac{(x-k\widehat{a})^2}{2\widehat{\sigma}^2 k} - \frac{(x-a(n-k))^2}{2\sigma^2(n-k)}\right\} + o\left(\frac{1}{\sqrt{k(n-k)}}\right), \end{aligned}$$

where

$$Q = \frac{\widehat{\mathbb{P}}^2(\tau = \infty)\mathbb{P}(\overline{\tau}_1 = \infty)}{\widehat{\sigma}\sigma} \sum_{y=1}^{\infty} \mathbb{P}(\overline{\tau}_y = \infty)\mathbb{P}(X_1 \le -y). \tag{3.51}$$

3.4.2 Proof of Corollary 3.2

We first prove (3.5). Using (3.45), we obtain

$$\begin{aligned} \mathbf{P}(\theta_\tau = k, M_\tau = x) &= \mathbf{P}(S_k = x, M_{k-1} < x, \tau > k)\mathbf{P}(\tau_+ > \tau_x) \\ &= e^{-\lambda x}\widehat{\mathbf{P}}(S_k = x, M_{k-1} < x, \tau > k)\mathbf{P}(\tau_+ > \tau_x). \end{aligned}$$

It is obvious that $\mathbf{P}(\tau_+ > \tau_x) \sim \mathbf{P}(\tau_+ = \infty)$ as $x \to \infty$. From this relation and from Theorem 3.8 with $y = z = 0$ we have

$$e^{\lambda x}\mathbf{P}(\theta_\tau = k, M_\tau = x) = \frac{c}{\sqrt{k}}e^{-(x-\widehat{a}k)^2/2\widehat{\sigma}^2 k} + o\left(\frac{1}{\sqrt{k+x}}\right)$$
$$= \frac{c\sqrt{\widehat{a}}}{\sqrt{x}}e^{-\widehat{a}^3(k-x/\widehat{a})^2/2\widehat{\sigma}^2 x} + o\left(\frac{1}{\sqrt{k+x}}\right).$$

Combining this expression with (3.2), we conclude that, uniformly in k,

$$\sqrt{x}\mathbf{P}(\theta_\tau = k | M_\tau = x) = \frac{c\sqrt{\widehat{a}}}{c_0}e^{-\widehat{a}^3(k-x/\widehat{a})^2/2\widehat{\sigma}^2 x} + o(1).$$

Summing over all k satisfying $|k - x/\widehat{a}| \le A\sqrt{x}$ and letting $A \to \infty$, we obtain

$$\lim_{A\to\infty}\lim_{x\to\infty}\mathbf{P}(|\theta_\tau - x/\widehat{a}| \le A\sqrt{x} | M_\tau = x) = \frac{c\sqrt{2\pi\widehat{\sigma}^2}}{c_0\widehat{a}}.$$

It remains to note that (3.8) implies that the left hand side in the previous relation equals 1. Therefore, $c = c_0\widehat{a}/\sqrt{2\pi\widehat{\sigma}^2}$ and (3.5) is proven.
By the total probability formula,

$$\mathbf{P}(\tau - \theta_\tau = j, M_\tau = x) = \sum_{k=1}^{\infty}\mathbf{P}(\theta_\tau = k, \tau = k + j, M_\tau = x)$$
$$= \sum_{k=1}^{\infty}\mathbf{P}(S_k = x, M_{k-1} < x, \tau > k)\mathbf{P}(\tau_+ > \tau_x = j)$$

and

$$\mathbf{P}(M_\tau = x) = \sum_{k=1}^{\infty}\mathbf{P}(S_k = x, M_{k-1} < x, \tau > k)\mathbf{P}(\tau_+ > \tau_x).$$

Therefore, uniformly in j,

$$\mathbf{P}(\tau - \theta_\tau = j | M_\tau = x) = \frac{\mathbf{P}(\tau_+ > \tau_x = j)}{\mathbf{P}(\tau_+ > \tau_x)}$$
$$\sim \frac{\mathbf{P}(\tau_+ > \tau_x = j)}{\mathbf{P}(\tau_+ = \infty)}, \quad x \to \infty. \tag{3.52}$$

Furthermore,

$$\mathbf{P}(\tau_+ > \tau_x = j) = \mathbf{P}(\overline{M}_j \ge x, \overline{M}_{j-1} < x, \overline{\tau}_1 > j).$$

If $j \le x/2a$ then, by the Kolmogorov inequality,

$$\mathbf{P}(\tau_+ > \tau_x = j) \le \mathbf{P}(\overline{M}_j \ge x) = O\left(\frac{1}{x}\right). \tag{3.53}$$

Therefore, it remains to prove (3.6) for $j > x/2a$. For such values of j we shall use the following representation:

$$\begin{aligned}
\mathbf{P}(\tau_+ > \tau_x = j) &= \mathbf{P}(\overline{M}_j \geq x, \overline{M}_{j-1} < x, \overline{\tau}_1 > j) \\
&= \sum_{y=1}^{x} \mathbf{P}(\overline{M}_{j-1} < x, \overline{S}_{j-1} = x - y, \overline{\tau}_1 > j-1)\mathbf{P}(X_1 \leq -y).
\end{aligned} \tag{3.54}$$

Fix some $N \geq 1$. Using (3.18), we obtain

$$\begin{aligned}
&\sum_{y=N}^{x} \mathbf{P}(\overline{M}_{j-1} < x, \overline{S}_{j-1} = x - y, \overline{\tau}_1 > j-1)\mathbf{P}(X_1 \leq -y) \\
&\qquad \leq \sum_{y=N}^{x} \mathbf{P}(\overline{S}_{j-1} = x - y)\mathbf{P}(X_1 \leq -y) \\
&\qquad \leq \frac{c_1}{\sqrt{j-1}} \sum_{y=N}^{x} \mathbf{P}(X_1 \leq -y) \leq \frac{c_1}{\sqrt{x}} \sum_{y=N}^{\infty} \mathbf{P}(X_1 \leq -y).
\end{aligned} \tag{3.55}$$

Applying Theorem 3.8 to the random walk $\{\overline{S}_n\}$, we have

$$\begin{aligned}
&\sum_{y=1}^{N-1} \mathbf{P}(\overline{M}_{j-1} < x, \overline{S}_{j-1} = x - y, \overline{\tau}_1 > j-1)\mathbf{P}(X_1 \leq -y) \\
&\quad = \frac{\mathbf{P}(\overline{\tau}_1 = \infty)}{\sqrt{2\pi\sigma^2 j}} e^{-(x-aj)^2/2\sigma^2 j} \sum_{y=1}^{N-1} \mathbf{P}(\overline{\tau}_y = \infty)\mathbf{P}(X_1 \leq -y) + o\left(\frac{1}{\sqrt{x+j}}\right).
\end{aligned} \tag{3.56}$$

Combining (3.54)-(3.56) and letting $N \to \infty$, we arrive at the relation

$$\begin{aligned}
\mathbf{P}(\tau_+ > \tau_x = j) &= \frac{c}{\sqrt{2\pi\sigma^2 j}} e^{-(x-aj)^2/2\sigma^2 j} + o\left(\frac{1}{\sqrt{x+j}}\right) \\
&= \frac{c\sqrt{a}}{\sqrt{2\pi\sigma^2 x}} e^{-a^3(j-x/a)^2/2\sigma^2 x} + o\left(\frac{1}{\sqrt{x}}\right), \quad j \geq x/2a.
\end{aligned} \tag{3.57}$$

Plugging (3.57) into (3.52) and taking into account (3.53) we conclude that, uniformly in j,

$$\mathbf{P}(\tau - \theta_\tau = j | M_\tau = x) = \frac{c'}{\sqrt{2\pi\sigma^2 x}} e^{-a^3(j-x/a)^2/2\sigma^2 x} + o\left(\frac{1}{\sqrt{x}}\right).$$

Thus, it remains to show that $c' = a^{3/2}$. It suffices to repeat the argument from the proof of (3.5) and to notice that

$$\begin{aligned}
&\lim_{A\to\infty} \lim_{x\to\infty} \mathbf{P}(|\tau - \theta_\tau - x/a| > A\sqrt{x} | M_\tau = x) \\
&\qquad \leq \lim_{A\to\infty} \lim_{x\to\infty} \frac{\mathbf{P}(|\tau_x - x/a| > A\sqrt{x})}{\mathbf{P}(\tau_+ = \infty)} = 0,
\end{aligned}$$

which follows from (3.52) and from the Kolmogorov inequality. This completes the proof of (3.6).

4 Tail asymptotics for the area under the excursion of a random walk with heavy-tailed increments

This section is the subject of the article [9] written in collaboration with Denis Denisov and Vitali Wachtel.

We continue studying tail behaviour of the distribution of the area under the positive excursion of a random walk which has negative drift but in this section incremets are heavy-tailed. We determine the asymptotics for tail probabilities for the area.

4.1 Introduction and statement of results

Let $\{S_n; n \geq 1\}$ be a random walk with i.i.d. increments $\{X_k; k \geq 1\}$. We shall assume that the increments have negative expected value, $\mathbf{E}X_1 = -a$. Let $\overline{F}(x) = \mathbb{P}(X_1 > x)$ be the tail distribution function of X_1. Let

$$\tau := \min\{n \geq 1 : S_n \leq 0\}$$

be the first time the random walk exits the positive half-line. We consider the area under the random walks excursion $\{S_1, S_2, \ldots, S_{\tau-1}\}$:

$$A_\tau := \sum_{k=0}^{\tau-1} S_k.$$

Since τ is finite almost surely, the area A_τ is finite as well. In this note we will study asymptotics for $\mathbb{P}(A_\tau > x)$, as $x \to \infty$, in the case when distribution of increments is heavy-tailed. This section continues the research of the section 2, where the light-tailed case has been considered.

The heavy-tailed asymptotics for $\mathbb{P}(A_\tau > x)$ was studied previously by Borovkov, Boxma and Palmowski [4]. They considered the case when the increments of the random walk have a distribution with regularly varying tail, that is $\overline{F}(x) = x^{-\alpha}L(x)$, where $L(x)$ is a slowly varying function. For $\alpha > 1$ they showed

$$\mathbb{P}(A_\tau > x) \sim \mathbf{E}\tau\overline{F}(\sqrt{2ax}), \quad x \to \infty. \tag{4.1}$$

These asymptotics can be explained by a traditional heavy-tailed one big jump heuristics. In order to have a huge area, the random walk should have a large jump, say y, at the very beginning of the excursion. After this jump the random walk goes down along the line $y - an$ according to the Law of Large Numbers. Thus, the duration of the excursion should approximately be around y/a. As a result, the area will be of order $y^2/2a$. Now, from the equality $x = y^2/2a$ one infers that a jump of order $\sqrt{2ax}$ is needed. Since the same strategy is valid for the maximum $M_\tau := \max_{n<\tau} S_n$ of the first excursion, one can rewrite (4.1) in the following way:

$$\mathbf{P}(A_\tau > x) \sim \mathbf{P}(M_\tau > \sqrt{2ax}), \quad x \to \infty.$$

However, the class of regularly varying distributions does not include all subexponential distributions and excludes, in particular, log-normal distribution and Weibull distribution with parameter $\beta < 1$. The asymptotics for these remaining cases have been put as an open problem in [25, Conjecture 2.2] for a strongly related workload process. We will reformulate this conjecture as follows

$$\mathbb{P}(A_\tau > x) \sim \mathbb{P}\left(\tau > \sqrt{\frac{2x}{a}}\right), \quad x \to \infty, \tag{4.2}$$

when $F \in \mathcal{S}$ and $\mathcal{S}$ is a subclass of subexponential distributions. Note that using the asymptotics for

$$\mathbb{P}(\tau > x) \sim \mathbf{E}\tau\overline{F}(ax) \tag{4.3}$$

from [10] for Weibull distributions with parameter $\beta < 1/2$, one can see that in this case asymptotics (4.2) is equivalent to (4.1). In this note we partially settle (4.2). It is not difficult to show that the same arguments hold for the workload process and to prove the same asymptotics for the area of the workload process, thus settling the original [25, Conjecture 2.2]. In passing we note that it is doubtful that (4.2) completely holds. The reason for that is that for both τ and A_τ the asymptotics (4.3) and (4.2) are no longer valid for Weibull distributions with parameter $\beta > 1/2$. The analysis for $\beta > 1/2$ involves more complicated optimization procedure leading to a Cramér series and it is unlikely that the answers will be the same for the area and for the exit time.
We will now present the results. We will start with the regularly varying case. In this case the connection between the tails of A_τ and M_τ is strong and we will be able to use the asymptotics for $\mathbb{P}(M_\tau > x)$ found in [16], see also a short proof in [6], to find the asymptotics for $\mathbb{P}(A_\tau > x)$.

Proposition 4.1. *We have the following two statements.*

(a) *If $\overline{F}(x) := \mathbf{P}(X_1 > x) = x^{-\alpha}L(x)$ with some $\alpha \ge 1$ and $\mathbf{E}|X_1| < \infty$ then, uniformly in $y \in [\varepsilon\sqrt{x}, \sqrt{2ax}]$,*

$$\mathbf{P}(A_\tau > x, M_\tau > y) \sim \mathbf{E}\tau\overline{F}(\sqrt{2ax}). \tag{4.4}$$

(b) *If $\overline{F}(x) \sim x^{-\varkappa}e^{-g(x)}$, where $g(x)$ is a monotone continuously differentiable function satisfying $\frac{g(x)}{x^\beta} \downarrow$ for $\beta \in (0, 1/2)$, and $\mathbf{E}|X_1|^\varkappa < \infty$ for some $\varkappa > 1/(1-\beta)$ then (4.4) holds uniformly in $y \in \left[\sqrt{2ax} - \frac{R\sqrt{2ax}}{g(\sqrt{2ax})}, \sqrt{2ax}\right]$.*

This statement implies obviously the following lower bound for the tail of A_τ:

$$\liminf_{x\to\infty} \frac{\mathbb{P}(A_\tau > x)}{\overline{F}(\sqrt{2ax})} \ge 1. \tag{4.5}$$

Furthermore, using this proposition, one can give an alternative proof of (4.1) under the assumption of the regular variation of $\overline{F}$, which is much simpler than the original one in [4]. We first split the event $\{A_\tau > x\}$ into two parts

$$\{A_\tau > x\} = \{A_\tau > x, M_\tau > y\} \cup \{A_\tau > x, M_\tau \le y\}.$$

Clearly,

$$\{A_\tau > x, M_\tau \le y\} \subseteq \{\tau > x/y\},$$

and therefore,

$$\mathbb{P}(A_\tau > x, M_\tau > y) \le \mathbb{P}(A_\tau > x) \le \mathbb{P}(A_\tau > x, M_\tau > y) + \mathbb{P}(\tau > x/y). \tag{4.6}$$

When $\alpha > 1$, according to Theorem I in Doney [13] or [10, Theorem 3.2],

$$\mathbb{P}(\tau > t) \sim \mathbf{E}\tau \bar{F}(at) \quad \text{as } t \to \infty.$$

Choosing $y = \varepsilon\sqrt{x}$ and recalling that $\overline{F}$ is regularly varying, we get

$$\mathbb{P}(\tau > x/y) = \mathbb{P}(\tau > \sqrt{x}/\varepsilon) \sim \varepsilon^\alpha \mathbf{E}\tau\overline{F}(\sqrt{x}). \tag{4.7}$$

It follows from the first statement of Proposition 4.1 that

$$\mathbf{P}(A_\tau > x, M_\tau > \varepsilon\sqrt{x}) \sim \mathbf{E}\tau\overline{F}(\sqrt{2ax}).$$

Plugging this and (4.7) into (4.6), we get, as $x \to \infty$,

$$\mathbf{E}\tau\overline{F}(\sqrt{2ax})(1 + o(1)) \le \mathbf{P}(A_\tau > x) \le \mathbf{E}\tau\overline{F}(\sqrt{2ax})\left(1 + \frac{\varepsilon^\alpha}{(2a)^{\alpha/2}} + o(1)\right).$$

Letting $\varepsilon \to 0$, we arrive at (4.1).
The case of semi-exponential distributions is more complicated. In particular it seems that in this case there is a regime when the asymptotics (4.1) are no longer valid. We will treat this case by using the exponential bounds similar to Section 2.4 of this work and asymptotics for $\mathbb{P}(\tau > x)$ from [10] and [7].
First we will introduce a sublclass of subexponential distributions that we will consider. We will assume that $\mathbf{E}[X_1^2] = \sigma^2 < \infty$. Without loss of generality we may assume that $\sigma = 1$. Let

$$\overline{F}(x) \sim e^{-g(x)}x^{-2}, \quad x \to \infty, \tag{4.8}$$

where $g(x)$ is an eventually increasing function such that eventually

$$\frac{g(x)}{x^{\gamma_0}} \downarrow 0, \quad x \to \infty, \tag{4.9}$$

for some $\gamma_0 \in (0,1)$. Due to the asymptotic nature of equivalence in (4.8) without loss of generality we may assume that g is continuously differentiable and that (4.9) hold for all $x > 0$. Clearly, monotonicity in (4.9) implies

$$g'(x) \le \gamma_0 \frac{g(x)}{x} \tag{4.10}$$

for all sufficiently large x. Using the Karamata representation theorem one can show that this class of subexponential distributions includes regularly varying distributions $\overline{F}(x) \sim x^{-r}L(x)$, for $r > 2$. Also, it is not difficult to show that lognormal distributions and Weibull distributions $(\overline{F}(x) \sim e^{-x^\beta}, \beta \in (0,1))$ belong to our class of distributions. Previously this class appeared in [31] for the analysis of large deviations of sums of subexponential random variables on the whole axis.
Now we are able to give rough (logarithmic) asymptotics for $\gamma_0 \le 1$.

Theorem 4.2. *Let* $\mathbf{E}[X_1] = -a < 0$ *and* $Var(X_1) < \infty$. *Assume that the distribution function* F *of* X_j *satisfies* (4.8) *and that* (4.9) *holds with* $\gamma_0 = 1$. *Then, there exists a constant* $C > 0$ *such that*

$$\mathbb{P}(A_\tau > x) \le Cx^{1/4} \exp\left\{ -g(\sqrt{2ax})\sqrt{1 - \frac{2Cg(\sqrt{2ax})}{a\sqrt{2ax}}} \right\}.$$

Furthermore, for any $\varepsilon > 0$ *there exist* $C > 0$ *such that,*

$$\liminf_{x\to\infty} \frac{\mathbb{P}(A_\tau > x)}{\overline{F}(\sqrt{2ax} + Cx^{1/4+\varepsilon})} \ge \mathbf{E}\tau.$$

In particular, if $\gamma_0 < 1$ *then*

$$\lim_{x\to\infty} \frac{\ln \mathbb{P}(A_\tau > x)}{\ln \overline{F}(\sqrt{2ax})} = 1.$$

To obtain the exact asymptotics we will impose a further assumption

$$xg'(x) \to \infty, \quad x \to \infty. \tag{4.11}$$

This assumption implies that

$$\frac{g(x)}{\log x} \to \infty. \tag{4.12}$$

In particular, it excludes all regularly varying distributions.

Theorem 4.3. *Let* $\mathbf{E}[X_1] = -a < 0$ *and* $\mathbf{Var}(X_1) < \infty$. *Assume that the distribution function* F *of* X_j *satisfies* (4.8), *that* (4.9) *holds with* $\gamma_0 < 1/2$ *and that* (4.11) *holds. Then,*

$$\mathbb{P}(A_\tau > x) \sim \mathbf{E}\tau\overline{F}(\sqrt{2ax}), \quad x \to \infty.$$

In this note we provided exact asymptotics for the case $\gamma_0 < 1/2$. We believe that this restriction is not technical and the asymptotics for $\gamma_0 \ge 1/2$ is different. This boundary is well-known, for example, the same bound appears in the analysis of the exact asymptotics for $\mathbb{P}(\tau > n)$ and $\mathbb{P}(S_n > an)$, see, correspondingly [10] and [7].
The conjecture in [25] was formulated for the workload process of a single-server queue rather than the area under the random walk excursion. However, one can prove analogous results for the Lévy processes by essentially the same arguments. It is well-known that the workload of the M/G/1 queue can be represented as a Lévy process and thus our results can be transferred to this setting almost immediately. We believe that the treatment of the workload of the general G/G/1 queue is not that different as well.

4.2 Proof of Proposition 4.1

Before giving the proof we will collect some known results that we will need in this and the following Sections. We will require the following statement, the first part of which follows from Theorem 2 in Foss, Palmowski and Zachary [16] (see also [6] for a short proof) and the second part from [10, Theorem 3.2].

Proposition 4.4. *Let* $\mathbf{E}[X_1] = -a$ *and either (a)* $\overline{F}(x) := \mathbf{P}(X_1 > x) = x^{-\alpha}L(x)$ *with some* $\alpha > 1$ *or (b)* $\overline{F}(x) \sim x^{-\varkappa}e^{-g(x)}$*, where* $g(x)$ *is a monotone continuously differentiable function satisfying* $\frac{g(x)}{x^\beta} \downarrow$ *for* $\beta \in (0, 1/2)$*, and* $\mathbf{E}|X_1|^\varkappa < \infty$ *for some* $\varkappa > 1/(1-\beta)$ *then for any fixed* k*,*

$$\mathbb{P}(M_k > y) \sim \mathbb{P}(S_k > y) \sim k\overline{F}(y), \quad y \to \infty \tag{4.13}$$

$$\mathbf{P}\left(\max_{n \le \tau \wedge k} S_n > y\right) \sim \mathbf{E}(\tau \wedge k)\overline{F}(y), \quad y \to \infty \tag{4.14}$$

$$\mathbb{P}(M_\tau > y) \sim \mathbf{E}\tau\bar{F}(y), \quad y \to \infty \tag{4.15}$$

and

$$\mathbb{P}(\tau > n) \sim \mathbf{E}[\tau]\overline{F}(an), \quad n \to \infty. \tag{4.16}$$

Proof.
To prove (4.13), (4.14) and (4.15), by Theorem 2 of [16] it is sufficient to show that (a) or (b) implies that $F \in \mathcal{S}^*$, that is $\int_0^\infty \overline{F}(y) < \infty$ and

$$\int_0^x \overline{F}(y)\overline{F}(x-y)dy \sim 2\overline{F}(x)\int_0^\infty \overline{F}(y)dy, \quad x \to \infty.$$

The fact that (a) implies $F \in \mathcal{S}^*$ is well-known and follows immediately from the dominated confergence theorem, since $\overline{F}(x) \sim \overline{F}(x-y)$ for all fixed y and

$$\int_0^x \frac{\overline{F}(y)\overline{F}(x-y)}{\overline{F}(x)}dy = 2\int_0^{x/2} \frac{\overline{F}(y)\overline{F}(x-y)}{\overline{F}(x)}dy$$

and $\overline{F}(x-y) \le C\overline{F}(x)$ for some $C > 0$ when $y \le x/2$.
Now, assume that (b) holds and show that $F \in \mathcal{S}^*$. Consider now

$$2\int_0^{x/2} \frac{\overline{F}(y)\overline{F}(x-y)}{\overline{F}(x)}dy.$$

Uniformly in $y \in [\ln x, x/2]$ we have

$$\begin{aligned}
\frac{\overline{F}(y)\overline{F}(x-y)}{\overline{F}(x)} &\le Ce^{g(x)-g(x-y)-g(y)} = Ce^{\int_{x-y}^x g'(t)dt - g(y)} \le Ce^{\beta\int_{x-y}^x \frac{g(t)}{t}dt - g(x-y)} \\
&\le Ce^{\beta y\frac{g(x-y)}{x-y} - g(x-y)} \le Ce^{(\beta-1)g(x-y)} \to 0, \quad x \to \infty,
\end{aligned}$$

and therefore,

$$2\int_{\ln x}^{x/2} \frac{\overline{F}(y)\overline{F}(x-y)}{\overline{F}(x)}dy \to 0.$$

Next for $y \in [0, \ln x]$,

$$\begin{aligned}
1 \le \frac{\overline{F}(x-y)}{\overline{F}(x)} &\le \frac{\overline{F}(x-\ln x)}{\overline{F}(x)} \sim e^{g(x)-g(x-\ln x)} = e^{\int_{x-\ln x}^x g'(t)dt} \\
&\le e^{\beta\int_{x-\ln x}^x \frac{g(t)}{t}dt} \le e^{\beta\frac{g(x-\ln x)}{(x-\ln x)^\beta}\int_{x-\ln x}^x t^{\beta-1}dt} \le e^{C\frac{g(x-\ln x)}{(x-\ln x)^\beta}\frac{\ln x}{x^{1-\beta}}} \to 1,
\end{aligned}$$

which implies that $F \in \mathcal{S}^*$.
The proof of (4.16) is very similar and can be done by straightforward verification that (4.8) and (4.9) imply that conditions of Theorem 3.1 (and hence of Theorem 3.2) of [10] hold. □

Define

$$\sigma_y = \inf\{n < \tau : S_n > y\}.$$

Then, for every $k \geq 1$,

$$\begin{aligned}\mathbb{P}(\sigma_y = k | M_\tau > y) &= \frac{\mathbb{P}(\sigma_y = k)}{\mathbb{P}(M_\tau > y)}\\ &= \frac{\mathbf{P}\left(\max_{n \leq \tau \wedge k} S_n > y\right) - \mathbf{P}\left(\max_{n \leq \tau \wedge (k-1)} S_n > y\right)}{\mathbf{P}(M_\tau > y)}.\end{aligned}$$

It follows from (4.14) and (4.15) that

$$\begin{aligned}\lim_{y\to\infty} \mathbb{P}(\sigma_y = k | M_\tau > y) &= \frac{\mathbf{E}\tau \wedge k - \mathbf{E}\tau \wedge (k-1)}{\mathbf{E}\tau}\\ &= \frac{\mathbf{P}(\tau > k-1)}{\mathbf{E}\tau} =: q_k, \quad k \geq 1.\end{aligned} \tag{4.17}$$

It is clear that

$$\sum_{k=1}^{\infty} q_k = \frac{1}{\mathbf{E}\tau} \sum_{k=0}^{\infty} \mathbb{P}(\tau > k-1) = 1.$$

For every fixed $N \geq 1$ we have

$$\begin{aligned}&\mathbb{P}(A_\tau > x, M_\tau > y)\\ &= \sum_{k=1}^{N} \mathbb{P}(A_\tau > x, \sigma_y = k, M_\tau > y) + \mathbb{P}(A_\tau > x, \sigma_y > N, M_\tau > y).\end{aligned} \tag{4.18}$$

For the last term on the right hand side we have

$$\begin{aligned}\mathbb{P}(A_\tau > x, \sigma_y > N, M_\tau > y) &\leq \mathbb{P}(\sigma_y > N, M_\tau > y)\\ &= \mathbb{P}(M_\tau > y)\mathbb{P}(\sigma_y > N | M_\tau > y).\end{aligned}$$

It follows from (4.17) that $\mathbb{P}(\sigma_y > N | M_\tau > y) \to \sum_{j=N+1}^{\infty} q_j$, as $y \to \infty$. Then, using (4.15), we get

$$\mathbb{P}(A_\tau > x, \sigma_y > N, M_\tau > y) \leq \varepsilon_N \bar{F}(y), \tag{4.19}$$

where $\varepsilon_N \to 0$ as $N \to \infty$.
For every fixed k we have

$$\mathbb{P}(A_\tau > x, \sigma_y = k, M_\tau > y) = \mathbb{P}(A_\tau > x, \sigma_y = k).$$

Since $S_j \in (0, y)$ for all $j < k$, we obtain

$$\mathbb{P}(A_\tau > x, \sigma_y = k) \le \mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > x - (k-1)y, \sigma_y = k\right)$$

and

$$\mathbb{P}(A_\tau > x, \sigma_y = k) \ge \mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > x, \sigma_y = k\right).$$

By the Markov property, for every $z > 0$,

$$\mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > z, \sigma_y = k\right) = \int_y^\infty \mathbb{P}(S_k \in dv, \sigma_y = k)\mathbb{P}(A_\tau > z | S_0 = v).$$

Let $\varkappa \in (1/(1-\beta), 2)$ if $\overline{F}$ satisfies the conditions of the part (b) and let $\varkappa = 1$ in the case when $\overline{F}$ is regularly varying. Fix some $\delta > 0$ and consider the set

$$B_v := \left\{v - \delta v^{1/\varkappa} \le S_n + na \le v + \delta v^{1/\varkappa} \text{ for all } n \le \frac{v + \delta v^{1/\varkappa}}{a}\right\}.$$

Since $\mathbf{E}|X_1|^\varkappa < \infty$, it follows from the Marcinkiewicz-Zygmund Law of Large Numbers that

$$\mathbf{P}(B_v | S_0 = v) \to 1 \quad \text{as } v \to \infty. \tag{4.20}$$

This implies that, as $y \to \infty$,

$$\mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > z, \sigma_y = k\right)$$
$$= \int_y^\infty \mathbb{P}\left(S_k \in dv, \sigma_y = k\right) \mathbb{P}\left(\{A_\tau > z\} \cap B_v | S_0 = v\right) + o\left(\mathbb{P}(\sigma_y = k)\right).$$

On the event B_v one has

$$\frac{(v - \delta v^{1/\varkappa})^2}{2a} \le A_\tau \le \frac{(v + \delta v^{1/\varkappa})^2}{2a}.$$

In other words,

$$\mathbb{P}\left(\{A_\tau > z\} \cap B_v | S_0 = v\right) = \mathbb{P}(B_v) \quad \text{if} \quad v - \delta v^{1/\varkappa} \ge \sqrt{2az}$$

and

$$\mathbb{P}\left(\{A_\tau > z\} \cap B_v | S_0 = v\right) = 0 \quad \text{if} \quad v + \delta v^{1/\varkappa} < \sqrt{2az}.$$

Therefore, for all v large enough,

$$\mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > z, \sigma_y = k\right) \le \int_{\sqrt{2az} - \delta(2az)^{1/2\varkappa}}^\infty \mathbb{P}(S_k \in dv, \sigma_y = k) + o(\mathbb{P}(\sigma_y = k))$$
$$= \mathbb{P}\left(S_{\sigma_y} > \sqrt{2az} - \delta(2az)^{1/2\varkappa}, \sigma_y = k\right) + o(\mathbb{P}(\sigma_y = k))$$

and

$$\mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > z, \sigma_y = k\right) \leq \int_{\sqrt{2az}+2\delta(2az)^{1/2\varkappa}}^{\infty} \mathbb{P}(S_k \in dv, \sigma_y = k)\mathbf{P}(B_v) + o(\mathbb{P}(\sigma_y = k))$$
$$= \mathbb{P}\left(S_{\sigma_y} > \sqrt{2az} + 2\delta(2az)^{1/2\varkappa}, \sigma_y = k\right) + o(\mathbb{P}(\sigma_y = k)).$$

Lemma 4.5. *For every fixed k,*

$$\sup_{v>y}\left|\frac{\mathbb{P}(S_k > v, \sigma_y = k)}{\overline{F}(v)} - \mathbb{P}(\tau > k-1)\right| \to 0 \quad \text{as } y \to \infty.$$

Proof.
Fix some $N > 0$ and define the events

$$D_{k,N} = \cup_{j=1}^{k} \{X_j > v + kN, |X_l| \leq N \ \text{ for all } l \neq j, l \leq k\}.$$

It is clear that $D_{k,N} \subseteq \{S_k > v\}$. Therefore,

$$\begin{aligned}\mathbb{P}(S_k > v, \sigma_y = k) &= \mathbb{P}(D_{k,N}, \sigma_y = k) + \mathbb{P}(S_k > v, D^c_{k,N}, \sigma_y = k)\\ &= \mathbb{P}(X_k > v + kN, |X_l| \leq N, \text{for all } l < k, \sigma_y > k-1)\\ &\quad + \mathbb{P}(S_k > v, D^c_{k,N}, \sigma_y = k).\end{aligned}$$

For the first term we have ($y > (k-1)N$)

$$\begin{aligned}&\mathbb{P}(X_k > v + kN, |X_l| \leq N, \text{for all } l > k, \sigma_y > k-1)\\ &\quad= \mathbb{P}(\tau > k-1, |X_l| \leq N, l < k)\overline{F}(v + kN)\\ &\quad= \mathbb{P}(\tau > k-1)\overline{F}(v) - \varepsilon_N^{(1)}\overline{F}(v) + o(\overline{F}(v)), \quad \text{uniformly in } v > y,\end{aligned} \tag{4.21}$$

where

$$\varepsilon_N^{(1)} := \mathbb{P}(\tau > k-1, |X_l| > N \text{ for some } l < k) \to 0 \quad N \to \infty.$$

Furthermore,

$$\begin{aligned}&\mathbb{P}(S_k > v, D^c_{k,N}, \sigma_y = k) \leq \mathbb{P}(S_k > v, D^c_{k,N}) = \mathbb{P}(S_k > v) - \mathbb{P}(D_{k,N})\\ &\quad= \mathbb{P}(S_k > v) - k\mathbb{P}(X_1 > v + kN)(\mathbb{P}(|X_1| \leq N))^{k-1}\\ &\quad= \varepsilon_N^{(2)}\overline{F}(v) + o(\overline{F}(v)),\end{aligned} \tag{4.22}$$

where

$$\varepsilon_N^{(2)} := k\left(1 - (\mathbb{P}(|X_1| \leq N))^{k-1}\right) \to 0, \quad N \to \infty.$$

Combining (4.21) and (4.22) and letting $N \to \infty$ we set the desired relation. □

Since with the previous lemma

$$\mathbb{P}(S_{\sigma_y} > v, \sigma_y = k) \sim \bar{F}(v)\mathbb{P}(\tau > k-1), \quad v, y \to \infty$$

for $v \geq y$, we infer that

$$\mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > z, \sigma_y = k\right) \leq \bar{F}\left(\sqrt{2az} - \delta(2az)^{1/2\varkappa}\right)(\mathbb{P}(\tau > k-1) + o(1)) + o(\mathbf{P}(\sigma_y = k))$$

and

$$\mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > z, \sigma_y = k\right) \geq \overline{F}\left(\sqrt{2az} + 2\delta(2az)^{1/2\varkappa}\right)(\mathbb{P}(\tau > k-1) + o(1)) + o(\mathbf{P}(\sigma_y = k)).$$

Under our assumptions on $\overline{F}$ one has

$$\lim_{\delta\to 0}\lim_{z\to\infty} \frac{\overline{F}\left(\sqrt{2az} + 2\delta(2az)^{1/2\varkappa}\right)}{\overline{F}\left(\sqrt{2az} - \delta(2az)^{1/2\varkappa}\right)} = 1.$$

Therefore,

$$\mathbb{P}\left(\sum_{j=k}^{\tau-1} S_j > z, \sigma_y = k\right) = \overline{F}\left(\sqrt{2az}\right)(\mathbb{P}(\tau > k-1) + o(1)) + o(\mathbf{P}(\sigma_y = k)).$$

Consequently,

$$\mathbb{P}(A_\tau > x, \sigma_y = k) = \bar{F}(\sqrt{2ax})\mathbb{P}(\tau > k-1) + o\left(\mathbb{P}(\sigma_y = k)\right).$$

Combining (4.15) and (4.17), one gets

$$\mathbf{P}(\sigma_y = k) \sim q_k \mathbf{E}\tau\overline{F}(y).$$

Therefore,

$$\mathbb{P}(A_\tau > x, \sigma_y = k) = \bar{F}(\sqrt{2ax})(\mathbb{P}(\tau > k-1) + o(1)) + o(\overline{F}(y)).$$

Consequently,

$$\sum_{k=1}^{N} \mathbb{P}(A_\tau > x, \sigma_y = k, M_\tau > y) = (\overline{F}(\sqrt{2ax}) + o(1))\sum_{k=1}^{N} \mathbb{P}(\tau > k-1) + o(\overline{F}(y)). \tag{4.23}$$

Plugging (4.19) and (4.23) into (4.18) and letting $N \to \infty$, we obtain

$$\mathbb{P}(A_\tau > x, M_\tau > y) = (\mathbf{E}\tau + o(1))\bar{F}(\sqrt{2ax}) + o(\overline{F}(y)).$$

Thus, it remains to show that $\overline{F}(y) = O(\overline{F}(\sqrt{2ax}))$. This is obvious for regularly varying tails and $y \geq \varepsilon\sqrt{x}$.
Assume now that $\overline{F}$ satisfies the conditions of part (b). To simplify notation put $y_* = \sqrt{2ax} - \frac{R\sqrt{2ax}}{g(\sqrt{2ax})}$. Then,

$$1 \leq \frac{\overline{F}(y_*)}{\overline{F}(\sqrt{2ax})} \leq (1+o(1))e^{g(\sqrt{2ax})-g(y_*)}.$$

Since $\frac{g(x)}{x^\beta}$ is monotone decreasing and g is differentiable then clearly

$$g'(x) \leq \beta\frac{g(x)}{x}.$$

Then,

$$\begin{aligned}
g(\sqrt{2ax}) - g\left(y_*\right) &= \int_{y_*}^{\sqrt{2ax}} g'(t)dt \leq \beta\int_{y_*}^{\sqrt{2ax}} \frac{g(t)}{t}dt \leq \beta\frac{g(y_*)}{(y_*)^\beta}\int_{y_*}^{\sqrt{2ax}} \frac{dt}{t^{1-\beta}} \\
&= \frac{g(y_*)}{(y_*)^\beta}((2ax)^{\beta/2} - (y_*)^\beta) \leq \frac{g(y_*)}{(y_*)^\beta}\frac{\beta}{(y_*)^{1-\beta}}C\frac{\sqrt{2ax}}{g(\sqrt{2ax})} \\
&\leq \beta C\frac{\sqrt{2ax}}{y_*} \leq (1+o(1))\beta C.
\end{aligned}$$

Therefore,

$$\overline{F}(y) \leq C\overline{F}(x), \qquad \forall y \in \left[\sqrt{2ax} - \frac{R\sqrt{2ax}}{g(\sqrt{2ax})}, \sqrt{2ax}\right].$$

4.3 Proof of Theorem 4.2

We start by proving an exponential estimate for the area A_n when random variables X_j are truncated. Let

$$\overline{X}_n = \max(X_1, \dots, X_n).$$

The next result is our main technical tool to investigate trajectories without big jumps.

Lemma 4.6. *Let $\mathbf{E}[X_1] = -a$ and $\sigma^2 := \mathbf{Var}(X_1) < \infty$. Assume that the distribution function F of X_j satisfies (4.8) and that (4.9) holds with $\gamma_0 = 1$. Then, there exists a constant $C > 0$ such that*

$$\mathbb{P}(A_n > x, \overline{X}_n \leq y) \leq \exp\left\{-\lambda\frac{x}{n} - \lambda\frac{an}{2} + C\lambda^2 n\right\},$$

where $\lambda = \frac{g(y)}{y}$.

Proof.
We will prove this Lemma by using the exponential Chebyshev inequality. For that we need to obtain estimates for the moment generating function of A_n. First,

$$\mathbf{E}\left[e^{\frac{\lambda}{n}A_n}; \overline{X}_n \leq y\right] = \mathbf{E}\left[e^{\frac{\lambda}{n}\sum_1^n (n-j+1)X_j}; \overline{X}_n \leq y\right] = \prod_{j=1}^n \varphi_y\left(\lambda_{n,j}\right),$$

where
$$\varphi_y(t) := \mathbf{E}[e^{tX_j}; X_j \le y]$$
and
$$\lambda_{n,j} := \lambda \frac{(n-j+1)}{n}.$$
Then,
$$\begin{aligned}\varphi_y(\lambda_{n,j}) &= \mathbf{E}[e^{\lambda_{n,j}X_j}; X_j \le 1/\lambda_{n,j}] + \mathbf{E}[e^{\lambda_{n,j}X_j}; 1/\lambda_{n,j} < X_j \le y] \\ &=: E_1 + E_2.\end{aligned}$$
Using the elementary bound $e^x \le 1 + x + x^2$ for $x \le 1$ we obtain,
$$E_1 \le 1 + \lambda_{n,j}\mathbf{E}[X_j] + \lambda_{n,j}^2\mathbf{E}[X_j^2] = 1 - a\lambda_{n,j} + (a^2+\sigma^2)\lambda_{n,j}^2.$$
Next, using the integration by parts and the assumption (4.8),
$$\begin{aligned}E_2 &= \int_{1/\lambda_{n,j}}^{y} e^{\lambda_{n,j}t} dF(t) = -\overline{F}(t)e^{\lambda_{n,j}t}\Big|_{t=1/\lambda_{n,j}}^{t=y} + \lambda_{n,j}\int_{1/\lambda_{n,j}}^{y} e^{\lambda_{n,j}t}\overline{F}(t)dt \\ &\le e\overline{F}(1/\lambda_{n,j}) + C\lambda_{n,j}\int_{1/\lambda_{n,j}}^{y} e^{\lambda_{n,j}t-g(t)}t^{-2}dt.\end{aligned}$$
Now note that for $t \le y$,
$$\lambda_{n,j}t - g(t) = t\left(\lambda_{n,j} - \frac{g(t)}{t}\right) \le t\left(\lambda_{n,j} - \frac{g(y)}{y}\right),$$
due to the condition (4.9). Then,
$$\lambda_{n,j} - \frac{g(y)}{y} \le \lambda - \frac{g(y)}{y} = 0$$
and, therefore,
$$E_2 \le e\overline{F}(1/\lambda_{n,j}) + C\lambda_{n,j}\int_{1/\lambda_{n,j}}^{y} t^{-2}dt \le (C+e)\lambda_{n,j}^2,$$
where we also used the Chebyshev inequality. As a result, for some constant C,
$$\varphi_y(t) = E_1 + E_2 \le 1 - a\lambda_{n,j} + C\lambda_{n,j}^2.$$
Consequently,
$$\begin{aligned}\mathbf{E}\left[e^{\frac{\lambda}{n}A_n}; \overline{X}_n \le y\right] &\le \prod_{j=1}^{n}\left(1 - a\lambda_{n,j} + C\lambda_{n,j}^2\right) \\ &= \exp\left\{\sum_{j=1}^{n}\ln\left(1 - a\lambda_{n,j} + C\lambda_{n,j}^2\right)\right\} \\ &\le \exp\left\{\sum_{j=1}^{n}\left(-a\lambda_{n,j} + C\lambda_{n,j}^2\right)\right\} \\ &= \exp\left\{\sum_{j=1}^{n}\left(-a\lambda\frac{n-j+1}{n} + C\left(\lambda\frac{n-j+1}{n}\right)^2\right)\right\} \\ &\le \exp\left\{-\frac{a\lambda}{2}n + C\lambda^2 n\right\}.\end{aligned}$$

Finally,

$$\mathbb{P}(A_n > x, \overline{X}_n \le y) \le e^{-\lambda\frac{x}{n}}\mathbf{E}\left[e^{\frac{\lambda}{n}A_n}; \overline{X}_n \le y\right] \le \exp\left\{-\lambda\frac{x}{n} - \frac{a\lambda}{2}n + C\lambda^2 n\right\}.$$

□

We can now obtain a rough upper bound using the exponential bound in Lemma 4.6.

Lemma 4.7. *Let* $\mathbf{E}[X_1] = -a < 0$ *and* $\mathbf{Var}(X_1) < \infty$*. Assume that the distribution function* F *of* X_j *satisfies* (4.8) *and that* (4.9) *holds with* $\gamma_0 = 1$*. Then, there exists a constant* $C > 0$ *such that*

$$\mathbb{P}(A_\tau > x) \le Cx^{1/4}\exp\left\{-g(\sqrt{2ax})\sqrt{1 - \frac{2Cg(\sqrt{2ax})}{a\sqrt{2ax}}}\right\}$$

.

Proof.
Clearly,

$$\mathbb{P}(A_\tau > x) \le \mathbb{P}(A_\tau > x, \overline{X}_\tau \le \sqrt{2ax}) + \mathbb{P}(A_\tau > x, \overline{X}_\tau > \sqrt{2ax}) =: P_1 + P_2.$$

First, using Lemma 4.6 with $y = \sqrt{2ax}$ we obtain,

$$\begin{aligned}P_1 &\le \sum_{n=0}^{\infty}\mathbf{P}(A_n \ge x, \overline{X}_n \le \sqrt{2ax}, \tau = n+1)\\ &\le \sum_{n=1}^{\infty}\exp\left\{-\lambda\frac{x}{n} - \frac{a\lambda}{2}n + C\lambda^2 n\right\} = \sum_{n=1}^{\infty}\exp\left\{-\lambda\frac{x}{n} - \lambda I n\right\},\end{aligned}$$

where $\lambda = \frac{g(\sqrt{2ax})}{\sqrt{2ax}}$ and $I = \frac{a}{2} - C\lambda$. With formula (25) at page 146 of Bateman [3] we have,

$$\begin{aligned}\sum_{n=1}^{\infty}\exp\left\{-\lambda\frac{x}{n} - \lambda I n\right\} &\le \int_0^\infty \exp\left\{-\lambda\frac{x}{y} - \lambda I(y+1)\right\}dy\\ &= e^{-\lambda I}\sqrt{\frac{4x}{I}}K_1(2\lambda\sqrt{Ix}).\end{aligned}$$

Now using the asymptotics for the modified Bessel function

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}$$

we obtain

$$\sum_{n=1}^{\infty}\exp\left\{-\lambda\frac{x}{n} - \lambda I n\right\} \le Cx^{1/4}\exp\{-2\lambda\sqrt{Ix}\}.$$

Therefore,

$$\begin{aligned} P_1 &\le Cx^{1/4}\exp\{-2\lambda\sqrt{Ix}\} \\ &\le Cx^{1/4}\exp\left\{-g(\sqrt{2ax})\sqrt{1-\frac{2Cg(\sqrt{2ax})}{a\sqrt{2ax}}}\right\}. \end{aligned} \tag{4.24}$$

Next,

$$\begin{aligned} P_2 &\le \sum_{n=0}^{\infty}\mathbf{P}(A_\tau \ge x, M_n \le \sqrt{2ax}, X_{n+1} > \sqrt{2ax}, \tau > n) \\ &\le \sum_{n=0}^{\infty}\mathbf{P}(X_{n+1} > \sqrt{2ax})\mathbb{P}(\tau > n) \le \mathbf{E}[\tau]\overline{F}(\sqrt{2ax}) = o(P_1). \end{aligned}$$

Then, the claim follows. □

Now we will give a lower bound.

Lemma 4.8. *Let* $\mathbf{E}[X_1] = -a < 0$ *and* $\mathbf{Var}(X_1) < \infty$. *Then, for any* $\varepsilon > 0$ *there exists* $C > 0$ *such that,*

$$\liminf_{x\to\infty}\frac{\mathbb{P}(A_\tau > x)}{\overline{F}(\sqrt{2ax}+Cx^{1/4+\varepsilon})} \ge \mathbf{E}\tau.$$

Proof.
Fix $N \ge 1$. Put $y^+ = \sqrt{2ax} + Cx^{1/2-\varepsilon}$, where C will be picked later. Since $\mathbf{E}[X_1^2] < \infty$, by the Strong Law of Large Numbers,

$$\frac{S_l + al}{l^{1/2+\varepsilon}} \to 0, \quad l \to \infty \text{ a.s.}$$

Hence, for any $\delta > 0$ we can pick $R > 0$ such that

$$\mathbb{P}\left(\min_{l\le\sqrt{2x/a}}(S_l + al + R + l^{1/2+\varepsilon}) > 0\right) > (1-\delta).$$

Now note that there exists a sufficiently large C such that, for every $k \le N$,

$$\left\{\min_{l\le\sqrt{2x/a}}(S_{k+l} - S_k + al + R + l^{1/2+\varepsilon}) > 0, \tau > k, S_k > y^+\right\} \subset \{A_\tau > x\}.$$

Hence,

$$\begin{aligned} &\mathbb{P}(A_\tau > x) \ge \sum_{k=0}^{N}\mathbb{P}(A_\tau > x, \overline{X}_{k-1} \le y^+, X_k > y^+, \tau > k) \\ &\ge \sum_{k=0}^{N}\mathbb{P}\left(\overline{X}_{k-1} \le y^+, \tau > k-1, X_k > y^+, \min_{l\le\sqrt{2x/a}}(S_{l+k} - S_k + R + j^{1/2+\varepsilon}) > 0\right) \\ &\ge (1-\delta)\sum_{k=0}^{N}\mathbb{P}\left(\overline{X}_{k-1} \le y^+, \tau > k-1\right)\overline{F}(y^+). \end{aligned}$$

For every fixed k we have

$$\mathbb{P}\left(\overline{X}_{k-1} \le y^+, \tau > k-1\right) \to \mathbb{P}\left(\tau > k-1\right), \quad x \to \infty.$$

Furthermore, $\sum_{k=0}^{N} \mathbb{P}(\tau > k) \to \mathbf{E}\tau$ as $N \to \infty$. Therefore, we can pick sufficiently large N such that

$$\liminf_{x\to\infty} \sum_{k=0}^{N} \mathbb{P}\left(\overline{X}_{k-1} \le y^+, \tau > k-1\right) \ge (1-\delta)\mathbf{E}\tau.$$

Then, for all x sufficiently large,

$$\mathbb{P}(A_\tau > x) \ge (1-\delta)^2 \mathbf{E}\tau \overline{F}(y^+).$$

As $\delta > 0$ is arbitrarily small we arrive at the conclusion. □

Completion of the proof of Theorem 4.2. The upper bound follows from Lemma 4.7. The lower bound follows from Lemma 4.8. The rough asymptotics follows immediately from the lower and upper bounds and from the observation that

$$\sup_{|y|\le x\rho(x)} \left| \frac{\log \overline{F}(x)}{\log \overline{F}(x+y)} - 1 \right| \to 0, \tag{4.25}$$

where $\rho(x) \to 0$.
To prove (4.25) we note that by (4.9) and (4.10)

$$\begin{aligned} g(x+y) - g(x) &= \int_x^{x+y} g'(t)dt \le \gamma_0 \int_x^{x+y} \frac{g(t)}{t} dt \le \gamma_0 \frac{g(x)}{x^{\gamma_0}} \int_x^{x+y} \frac{1}{t^{1-\gamma_0}} dt \\ &\le \gamma_0 \frac{g(x)}{x^{\gamma_0}} \frac{y}{x^{1-\gamma_0}} = \gamma_0 g(x) \frac{y}{x}, \quad y > 0. \end{aligned} \tag{4.26}$$

This implies that, as $x \to \infty$,

$$\sup_{|y|\le x\rho(x)} \left| \frac{g(x+y)}{g(x)} - 1 \right| \to 0. \tag{4.27}$$

Recalling that

$$\log \overline{F}(x) \sim -g(x) - 2\log x,$$

one obtains easily (4.25).

4.4 Proof of Theorem 4.3

Set

$$h(x) := \frac{\sqrt{2ax}}{g(\sqrt{2ax})}$$

and

$$y = \sqrt{2ax} - Ch(x)\log x, \tag{4.28}$$

where $C > \frac{5/4}{1-\gamma_0}$. First we will split the probability $\mathbb{P}(A_\tau > x)$ as follows

$$\mathbb{P}(A_\tau > x) = \mathbb{P}(A_\tau > x, \overline{X}_\tau \le y) + \mathbb{P}\left(A_\tau > x, \overline{X}_\tau > \sqrt{2ax} - \frac{1}{\log x}h(x)\right)$$
$$+ \mathbb{P}\left(A_\tau > x, \overline{X}_\tau \in \left[y, \sqrt{2ax} - \frac{1}{\log x}h(x)\right]\right) =: P_1 + P_2 + P_3.$$

The first term will be estimated using the exponential bound proved in Lemma 4.6.

Lemma 4.9. *Let* $\mathbf{E}[X_1] = -a$ *and* $\mathbf{Var}(X_1) < \infty$*. Assume that* (4.8) *and* (4.9) *hold for some* $\gamma_0 < 1/2$ *together with* (4.11)*. Then,*

$$P_1 = o(\overline{F}(\sqrt{2ax})).$$

Proof.
According to (4.24),

$$P_1 \le Cx^{1/4} \exp\{-2\lambda\sqrt{Ix}\},$$

where $I = \frac{a}{2} - C\lambda$ and $\lambda = g(y)/y$. Since (4.9) holds for some $\gamma_0 < 1/2$, $g^2(y)/y \to 0$ and hence

$$P_1 \le Cx^{1/4} \exp\left\{-\frac{g(y)}{y}\sqrt{2ax}\right\},$$

then

$$\frac{P_1}{\overline{F}(\sqrt{2ax})} \le Cx^{5/4} \exp\left\{g(\sqrt{2ax}) - \frac{g(y)}{y}\sqrt{2ax}\right\}.$$

To finish the proof it is sufficient to show that

$$g(\sqrt{2ax}) - \frac{g(y)}{y}\sqrt{2ax} + \frac{5}{4}\log x \to -\infty, \quad x \to \infty. \tag{4.29}$$

We first note that

$$d(x) := g(\sqrt{2ax}) - \frac{g(y)}{y}\sqrt{2ax} = g(\sqrt{2ax}) - \frac{g(y)}{1 - C\frac{\log x}{g(\sqrt{2ax})}}$$
$$= g(\sqrt{2ax}) - g(y) + (C + o(1))\log x \frac{g(y)}{g(\sqrt{2ax})}.$$

Using (4.10) and (4.9) one can see that

$$g(\sqrt{2ax}) - g(y) = \int_y^{\sqrt{2ax}} g'(z)dz \le \gamma_0 \int_y^{\sqrt{2ax}} \frac{g(z)}{z}dz \le \gamma_0 \frac{g(y)}{y}(\sqrt{2ax} - y) \tag{4.30}$$
$$= \gamma_0 C \frac{g(y)}{y} \log x \frac{\sqrt{2ax}}{g(\sqrt{2ax})}.$$

Hence,

$$d(x) \le \left(\gamma_0 \frac{\sqrt{2ax}}{y} - 1\right)(C + o(1))\frac{g(y)}{g(\sqrt{2ax})}\log x.$$

According to (4.27), $g(y) \sim g(\sqrt{2ax})$. Therefore, (4.29) is valid for any C satisfying $C(\gamma_0 - 1) + \frac{5}{4} < 0$.

□

Next lemma gives the term with the main contribution.

Lemma 4.10. *Under the assumptions of Lemma 4.9 we have the following estimate*

$$P_2 \le (1+o(1))\overline{F}(\sqrt{2ax}), \quad x \to \infty.$$

Proof.
Put

$$y^* = \sqrt{2ax} - \frac{h(x)}{\log x}.$$

By the total probability formula,

$$\begin{aligned} P_2 &\le \sum_{n=0}^{\infty} \mathbf{P}(A_\tau \ge x, \overline{X}_n \le y^*, X_{n+1} > y^*, \tau > n) \\ &\le \sum_{n=0}^{\infty} \mathbf{P}(X_{n+1} > y^*)\mathbb{P}(\tau > n) = \mathbf{E}[\tau]\overline{F}(y^*). \end{aligned}$$

Now note that by (4.30) and (4.27)

$$\begin{aligned} \frac{\overline{F}(y^*)}{\overline{F}(\sqrt{2ax})} &\le (1+o(1))e^{g(\sqrt{2ax})-g(y^*)} \le (1+o(1))e^{\frac{\gamma_0 g(y^*)}{y^*}(\sqrt{2ax}-y^*)} \\ &\le (1+o(1))e^{\frac{\gamma_0 g(y^*)}{y^*}\frac{1}{\log x}\frac{\sqrt{2ax}}{g(\sqrt{2ax})}} = 1+o(1). \end{aligned}$$

Then the statement immediately follows.

□

We will proceed to the analysis of P_3. Fix some $\delta > 0$ and set

$$z = \frac{1}{a}\left(\sqrt{2ax} + \delta\sqrt{x}\right).$$

We will split P_3 further as follows,

$$\begin{aligned} P_3 \le P_{31} + P_{32} + P_{33} &:= \mathbb{P}\left(A_\tau > x, \overline{X}_\tau \in \left[y, \sqrt{2ax} - R(x)h(x)\right]; J_1; \tau \le z\right) \\ &+ \mathbb{P}\left(A_\tau > x, \overline{X}_\tau \in \left[y, \sqrt{2ax} - R(x)h(x)\right]; J_{\ge 2}, \tau \le z\right) \\ &+ \mathbb{P}(\tau > z), \end{aligned}$$

where

$$J_1 = \left\{\text{there exists } k \in (1,\tau) \text{ such that } X_k > y \text{ and } \max_{1 \le i \le \tau, i \ne k} X_i \le y\right\}$$

and correspondingly,

$$J_{\ge 2} = \{\text{there exist } k, l \in (1,\tau) \text{ such that } X_k > y \text{ and } X_l > y\}.$$

We will start with easier terms P_{32} and P_{33}. To deal with these terms we will use Proposition 4.4. One can see then

Lemma 4.11. *Let the assumptions* (4.8),(4.9) *and* (4.11) *hold for* $\gamma_0 < 1/2$. *Then,*

$$P_{33} = o(\overline{F}(\sqrt{2ax})), \quad x \to \infty.$$

Proof.
We have, by Proposition 4.4,

$$P_{33} \le \mathbb{P}(\tau > z) \le (\mathbf{E}\tau + o(1))\overline{F}(az) = O\left(\overline{F}(\sqrt{2ax} + \delta\sqrt{x})\right).$$

Therefore,

$$\frac{P_{33}}{\overline{F}(\sqrt{2ax})} \le Ce^{g(\sqrt{2ax}) - g(\sqrt{2ax} + \delta\sqrt{x})}.$$

By the mean value theorem and by the assumption (4.11),

$$g(cx) - g(x) \to \infty, \quad x \to \infty$$

for every $c > 1$. This completes the proof. □

Lemma 4.12. *Let the conditions of Lemma 4.10 hold. Then,*

$$P_{32} = o(\overline{F}(\sqrt{2ax})). \tag{4.31}$$

Proof.
We can use the formula of total probability to write

$$P_{32} \le \sum_{k=1}^{z} \mathbb{P}(\tau > k, J_{\ge 2}) \le \sum_{k=1}^{z} \frac{k^2}{2}\overline{F}(y)^2.$$

Then,

$$\frac{P_{32}}{\overline{F}(\sqrt{2ax})} \le Cx^{3/2}\frac{\overline{F}(y)^2}{\overline{F}(\sqrt{2ax})} \le Cx^{1/2}e^{g(\sqrt{2ax}) - 2g(y)}.$$

Using now (4.30) one can see that

$$\frac{P_{32}}{\overline{F}(\sqrt{2ax})} \le Cx^{1/2}e^{C\ln x - g(y)} \to 0,$$

in view of (4.12). □

We are left to analyse P_{31}. For that, introduce

$$\mu(y) := \min\{n \ge 1 : X_k > y\}.$$

Now we will complete the proof with the following Lemma.

Lemma 4.13. *Let the assumptions* (4.8),(4.9) *and* (4.11) *hold for* $\gamma_0 < 1/2$. *Then,*

$$P_{31} = o(\overline{F}(\sqrt{2ax})), \quad x \to \infty.$$

Proof.
First represent event $J_1 = J_{11} \cup J_{12}$, where

$$J_{11} := \{X_k > y \text{ for exactly one } k \in (0,\tau) \text{ and } X_i \le x^\varepsilon \text{ for all other } i < \tau\ \}$$
$$J_{12} := \{X_k > y \text{ for exactly one } k \in (0,\tau) \text{ and } X_i > x^\varepsilon \text{ for some } i \ne k, i < \tau\}.$$

Then,

$$\begin{aligned} Q_2 &:= \mathbb{P}\left(A_\tau > x, \overline{X}_\tau \in \left[y, \sqrt{2ax} - \frac{1}{\log x} h(x)\right]; J_{12}, \tau \le z\right) \\ &\le \sum_{j=1}^{z} \mathbb{P}(\tau = j, J_{12}) \le \sum_{j=1}^{z} \frac{j^2}{2} \overline{F}(y)\overline{F}(x^\varepsilon) \le z^3 \overline{F}(y)\overline{F}(x^\varepsilon). \end{aligned}$$

Then,

$$\frac{Q_2}{\overline{F}(\sqrt{2ax})} \le C x^{3/2+2\varepsilon} e^{g(\sqrt{2ax}) - g(y) - g(x^\varepsilon))}$$

By (4.30),

$$g(\sqrt{2ax}) - g(y) \le C \ln x.$$

Then, in view of the relation (4.12) we have

$$g(\sqrt{2ax}) - g(y) - g(x^\varepsilon)) \le -4 \ln x,$$

which implies that $Q_2 = o(\overline{F}(\sqrt{2ax}))$.
To estimate

$$Q_1 := \mathbb{P}\left(A_\tau > x, \overline{X}_\tau \in \left[y, \sqrt{2ax} - \frac{1}{\log x} h(x)\right]; J_{11}, \tau \le z\right)$$

we make use of the exponential bound given in Lemma 4.6 putting

$$x^+(k) = x - k\left(\sqrt{2ax} - \frac{h(x)}{\log x}\right).$$

Then, we have,

$$\begin{aligned} Q_1 &= \sum_{k=0}^{z-1} \sum_{j=1}^{k} \mathbb{P}\left(A_k > x, \max_{i \ne j, i \le k} X_i \le x^\varepsilon, X_j \in \left[y, \sqrt{2ax} - \frac{h(x)}{\log x}\right], \tau = k+1\right) \\ &\le \sum_{k=1}^{z} (k+1) \mathbb{P}(A_k > x^+(k), \overline{X}_k \le x^\varepsilon) \overline{F}(y) \\ &\le C x^{1/2} \overline{F}(y) \sum_{k=1}^{z} \exp\left\{-\lambda \frac{x^+(k)}{k} - \frac{a\lambda}{2} k + C\lambda^2 k\right\}, \end{aligned}$$

where $\lambda = \frac{g(x^\varepsilon)}{x^\varepsilon}$. Now note that

$$-\lambda \frac{x^+(k)}{k} - \frac{a\lambda}{2} k = -\lambda\left(-\sqrt{2ax} + \frac{h(x)}{\log x} + \frac{x}{k} + \frac{ak}{2}\right).$$

Since

$$\frac{x}{k} + \frac{ak}{2} \geq \sqrt{2ax}, \quad k \geq 1,$$

we obtain,

$$-\lambda \frac{x^+(k)}{k} - \frac{a\lambda}{2} k \leq -\lambda \frac{h(x)}{\log x}, \quad k \geq 1.$$

Thus,

$$Q_1 \leq Cxe^{-\lambda h(x)/\log x + \lambda^2 z} \overline{F}(y).$$

Next, we can pick $\varepsilon = \frac{1}{4(1-\gamma_0)}$ to achieve

$$\begin{aligned}\lambda^2 z &\leq C \left(\frac{g(x^\varepsilon)}{x^\varepsilon}\right)^2 x^{1/2} = C\left(\frac{g(x^\varepsilon)}{x^{\varepsilon(1-1/(4\varepsilon))}}\right)^2 = C\left(\frac{g(x^\varepsilon)}{x^{\gamma_0 \varepsilon}}\right)^2 \\ &< C \sup_t \left(\frac{g(t)}{t^{\gamma_0}}\right)^2 < \infty,\end{aligned}$$

by the condition (4.9). Note that since $\gamma_0 < 1/2$, the picked $\varepsilon < 1/2$ as well. Then,

$$\frac{Q_1}{\overline{F}(\sqrt{2ax})} \leq Cx^2 e^{g(\sqrt{2ax}) - g(y) - \lambda h(x)/\log x},$$

and using (4.30),

$$\frac{Q_1}{\overline{F}(\sqrt{2ax})} \leq Cx^C e^{-\lambda h(x)/\log x}.$$

Finally noting that

$$\lambda h(x) = \frac{g(x^\varepsilon)}{x^\varepsilon} \frac{\sqrt{2ax}}{g(\sqrt{2ax})}$$

is decreasing polynomially we obtain required convergence to 0. The polynomial decay can be immediately seen for $g(x) = x^{\gamma_0}$. However, a proper proof goes as follows,

$$\begin{aligned}g(C\sqrt{x}) &= g(x^\varepsilon) + \int_{x^\varepsilon}^{C\sqrt{x}} g'(t)dt \leq g(x^\varepsilon) + \gamma_0 \int_{x^\varepsilon}^{C\sqrt{x}} \frac{g(t)}{t} dt \\ &\leq g(x^\varepsilon) + \gamma_0 \int_{x^\varepsilon}^{C\sqrt{x}} \frac{g(t)}{t^{\gamma_0}} t^{\gamma_0 - 1} dt \leq g(x^\varepsilon) + \frac{g(x^\varepsilon)}{x^{\varepsilon\gamma_0}} \int_{x^\varepsilon}^{C\sqrt{x}} t^{\gamma_0 - 1} dt \\ &\leq g(x^\varepsilon) + C\frac{g(x^\varepsilon)}{x^{\varepsilon\gamma_0}} x^{\gamma_0/2} \leq Cg(x^\varepsilon) x^{\gamma_0(1/2-\varepsilon)}.\end{aligned}$$

Therefore,

$$\lambda h(x) \geq x^{1/2-\varepsilon} x^{-\gamma_0(1/2-\varepsilon)}.$$

□

Completion of the proof of Theorem 4.3 Combination of the preceding Lemmas give us the upper bound. The lower bound has been shown in (4.5) under even weaker conditions.

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